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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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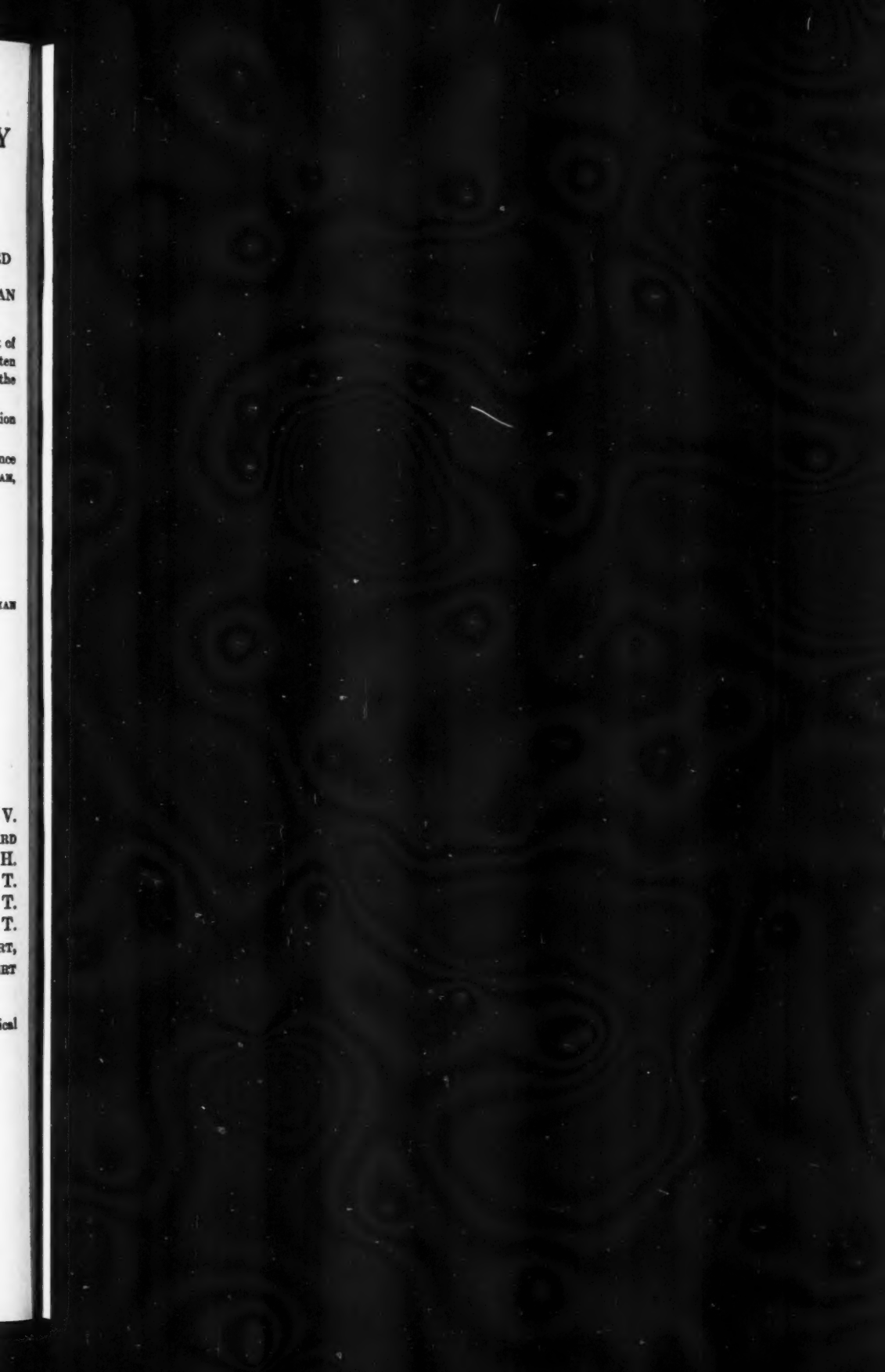

GENERALIZED FIBONACCI NUMBERS AND ASSOCIATED MATRICES

E. P. MILES, JR., The Florida State University

Introduction. We define k-generalized Fibonacci numbers ($k \geqq 2$) in such a way that for $k=2$ we get the ordinary Fibonacci numbers. We prove several interesting facts about these k-generalized numbers which reduce for $k=2$ to well-known properties of the ordinary Fibonacci numbers. We study a sequence of 2 by 2 nonsingular matrices with elements consecutive Fibonacci numbers whose members become arbitrarily ill-conditioned if we progress far enough in the sequence. This result is later generalized to obtain a sequence of k by k matrices with k-generalized Fibonacci numbers for elements and comparably ill-conditioned members.

The k-generalized Fibonacci numbers $f_{j,k}$ are defined as follows:

$$(1) \qquad f_{j,k} = 0, \quad 0 \leqq j \leqq k-2, \quad f_{k-1,k} = 1, \quad f_{jk} = \sum_{n=1}^{k} f_{j-n,k}, \quad j \geqq k.$$

When $k=2$ the numbers $f_{j,2}$, or simply f_j, which satisfy

$$(1)' \qquad f_0 = 0, \quad f_1 = 1, \quad f_j = f_{j-1} + f_{j-2}, \quad j > 1,$$

are the ordinary Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, $\cdots$. We first state without proof three well-known (see for instance [1]) properties (2), (3), (4) of these f_j. However, these properties are just special cases of their counterparts (2)″, (3)″, and (4)″ for the $f_{j,k}$ which we prove later. The first properties to be generalized are

$$(2) \qquad f_n = [(1+\sqrt{5})^n - (1-\sqrt{5})^n]/2^n\sqrt{5}, \quad n = 0, 1, \cdots,$$

$$(3) \qquad \lim_{n\to\infty} f_{n+1}/f_n = (1+\sqrt{5})/2,$$

$$(4) \qquad f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}, \quad n = 0, 1, \cdots.$$

It is convenient for recognizing the form which our generalizations will take to rewrite (2), (3), and (4) in terms of the concepts introduced in (5), (6), and (7) which follow. We consider the equation,

$$(5) \qquad E_2(x) = x^2 - x - 1 = 0,$$

having roots

$$(6) \qquad r_1 = (1-\sqrt{5})/2, \quad r_2 = (1+\sqrt{5})/2, \quad r_1 < r_2,$$

and the family of 2 by 2 matrices,

$$A_n = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} \qquad n = 0, 1, \cdots, \tag{7}$$

In terms of quantities just defined, (2), (3), and (4) become

$$f_n = r_1^n/(r_1 - r_2) + r_2^n/(r_2 - r_1), \tag{2}'$$

$$\lim_{n\to\infty} f_{n+1}/f_n = r_2, \tag{3}'$$

$$|A_n| = (-1)^{n+1}. \tag{4}'$$

Since the family of matrices A_n defined by (7) has some interesting properties not immediately apparent, we proceed with a development and discussion of these properties before passing to our generalizations. The matrix A_n is the nonsingular coefficient matrix for the system

$$f_n x_1 + f_{n+1} x_2 = f_{n+2}, \qquad f_{n+1} x_1 + f_{n+2} x_2 = f_{n+3}, \tag{8}$$

with the obvious unique solution (1, 1).

In view of (3) it is obvious that the system (8) becomes "nearly" dependent with increasing n, since the corresponding coefficients become almost proportional. This leads to excellent classroom illustrations of systems with integral coefficients and integral exact solutions which are highly unstable if the coefficients are known only approximately. For instance, consider the system (8) for $n=10$ under the assumption that all coefficients are exactly known except that of x_2 in the second equation which may have an error no greater than .02%. The system may be indicated as follows:

$$\begin{aligned} 55x_1 + 89x_2 &= 144, \\ 89x_1 + (144 \pm \epsilon)x_2 &= 233, \qquad 0 \leqq \epsilon \leqq .0288. \end{aligned} \tag{9}$$

For $\epsilon=0$ the correct solution is (1, 1), for $\epsilon=.02$ it becomes (18.8, -10), for $\epsilon=.018$ it is (-159.2, 100). This wide variation in the solutions for small changes in ϵ is to be expected since the system for $\epsilon=1/55$ is inconsistent and hence has no solution at all.

By using the P-condition number of von Neumann (absolute value of the ratio of larger eigenvalue of A_n to the smaller), we can make a precise statement about the ill-condition of A_n. The characteristic equation for A_n,

$$(f_n - \lambda)(f_{n+2} - \lambda) - f_{n+1}^2 = 0, \tag{10}$$

may, with the aid of (4), be written as

$$\lambda^2 - (f_n + f_{n+2})\lambda + (-1)^{n+1} = 0. \tag{11}$$

It is easily verified that A_0 has eigenvalues $\lambda_{1,0}=r_1=(1-\sqrt{5})/2$ and $\lambda_{2,0}=r_1$ $=(1+\sqrt{5})/2$ and P-condition number $=r_2/r_1=r_2^2=(3+\sqrt{5})/2$. Likewise A_1 has

eigenvalues $\lambda_{1,1}=(3-\sqrt{5})/2=r_1^2$ and $\lambda_{2,1}=(3+\sqrt{5})/2=r_2^2$ and P-condition number r_2^4. This suggests the general result which we now prove* that A_n has eigenvalues $\lambda_{1,n}=r_1^{n+1}$, $\lambda_{2,n}=r_2^{n+1}$ and P-condition number r_2^{2n+2}. From (5) we see that $r_1r_2=-1$. Then by (2)′ we have

$$
\begin{aligned}
f_n+f_{n+2} &= \frac{r_1^n}{r_1-r_2}+\frac{r_2^n}{r_2-r_1}+\frac{r_1^{n+2}}{r_1-r_2}+\frac{r_2^{n+2}}{r_2-r_1} \\
&= \frac{r_1^{n+1}}{r_1-r_2}\left(\frac{1}{r_1}+r_1\right)+\frac{r_2^{n+1}}{r_2-r_1}\left(\frac{1}{r_2}+r_2\right) \qquad (12) \\
&= \frac{r_1^{n+1}}{r_1-r_2}(-r_2+r_1)+\frac{r_2^{n+1}}{r_2-r_1}(-r_1+r_2) \\
&= r_1^{n+1}+r_2^{n+1}.
\end{aligned}
$$

Using (12), (11) becomes $\lambda^2-(r_1^{n+1}+r_2^{n+1})\lambda+r_1^{n+1}r_2^{n+1}=0$, which obviously has the roots specified. The P-condition number of A_n is thus seen to be r_2^{2n+2} which becomes arbitrarily large as n increases.

The conditions (1) completely determine the value of $f_{n,k}$. This is readily ascertainable from the general solution of the kth-order difference equation with constant coefficients,

$$a_0f(n+k)+a_1f(n+k-1)+\cdots+a_kf(n)=0, \qquad (13)$$

which has been known since Bernoulli (1728) (see, for instance, Aitken [2]). The general solution of (13) is of the form

$$f(n)=c_1z_1^n+\cdots+c_kz_k^n \qquad (14)$$

whenever the algebraic equation

$$a_0z^k+a_1z^{k-1}+\cdots+a_k=0 \qquad (15)$$

has k distinct roots $z_1, \cdots, z_k$. Substituting for n the values $0, 1, \cdots, k-1$ in (14) imposes k independent and consistent conditions on the constants $c_1, \cdots, c_k$ which determine them uniquely.

Thus we turn now to a study of the roots of

$$E_k''(x)=x^k-x^{k-1}-\cdots-x-1=0, \qquad (5)''$$

showing them to be distinct roots $r_{1,k}, \cdots, r_{k,k}$ which may be ordered so that

* An alternative proof involves showing that $A_0^*=A_{n-1}$, an interesting property noted by the author only after the first submission of the paper. This property, which for $k\geq 2$ does not apply to the $A_{n,k}$ of this paper, has been generalized in the author's Classroom Note, On matrix slide rules, pages 788–791 of this issue of the MONTHLY.

(6)″ $\qquad |r_{j,k}| < 1, \quad 1 \leqq j \leqq k-1 \quad \text{and} \quad 1 < r_{k,k} < 2.$

Since $E_k''(1) = 1-k < 0$ and $E_k''(2) = 1$, there is at least one root of $E_k''(x) = 0$ on the open interval (1, 2). Let the largest such root be called $r_{k,k}$. We now show that $E_k''(z)$ has no roots on the unit circle. In particular it does not have the root $z = 1$, so we may write (5)″ in equivalent form

$$z^k = (z^k - 1)/(z - 1), \qquad z \neq 1. \tag{16}$$

Assuming that (16), and hence (5)″, has a root $\cos\theta + i\sin\theta = z_1$ we obtain, by substituting z_1 in (16), equating the square of the absolute values of each side, and simplifying, the condition $\cos k\theta = \cos\theta$, which is satisfied only for $\theta = (2n\pi)/(k-1)$ or $\theta = (2n\pi)/(k+1)$ and integral n. The first value for θ makes $z_1^k = z_1 = 1$ which is already ruled out; the second value for θ makes z_1^k equal to $1/z_1$ and, by (16), to $-1/z_1$, which is impossible. Thus (5) has no roots on the unit circle. It is now helpful to introduce

$$E_k'(z) = z^{k+1} - 2z^k + 1 = 0, \tag{5′}$$

formed by multiplying (16) by $z-1$ and collecting terms. Obviously $E_k' = 0$ has k roots in common with $E_k'' = 0$. It has one additional root, $z = 1$, on the unit circle and at least one root $r_{k,k}$ outside the unit circle. We show that its roots inside the unit circle are distinct. Any repeated root of $E_k'' = 0$ would also be a root of $(k+1)z^k - 2kx^{k-1} = 0$ and accordingly either $z = 0$ or $z = (2k)/(k+1)$. Of these possibilities $z = 0$, the only one inside the unit circle, is not a root of $E_k' = 0$. We complete our analysis of the roots of $E_k'' = 0$ by showing that $E_k' = 0$ has exactly $k-1$ roots interior to the unit circle. We do this by showing that $E_k'(z)$ and $-2z^k + 1 = 0$ have the same number of zeros interior to the circle Γ_ϵ, Γ_ϵ: $z = (1+\epsilon)^{1/k}$, for sufficiently small positive ϵ. Let $f(z) = z^{k+1}$ and $g(z) = -2z^k + 1$. On Γ_ϵ we have $|f(z)| = (1+\epsilon)^{(k+1)/k}$ and $|g(z)| = |-2(1+\epsilon)e^{i\theta} + 1| > 2(1+\epsilon) - 1 = 1 + 2\epsilon$. Thus on Γ_ϵ,

$$|g(z)| / |f(z)| > [(1 + 2\epsilon)/[(1 + \epsilon)^{(k+1)/k}] = F(\epsilon).$$

Now, $F(0) = 1$ and

$$F'(0) = \frac{(1 + \epsilon)^{(k+1)/k}[2] - [1 + 2\epsilon][(k + 1)/k](1 + \epsilon)^{1/k}}{(1 + \epsilon)^{(2k+2)/k}}\Bigg|_{\epsilon=0},$$

so that $F'(0) = (k-1)/k > 0$. Thus for some positive ϵ^* sufficiently small we have $F(\epsilon) > 1$ for $0 < \epsilon \leqq \epsilon^*$.

Applying Rouche's theorem we see that, for all sufficiently small positive ϵ, $g(z) = -2z^k + 1$ and $f(z) + g(z) = E_k'(z)$ have the same number of zeros in Γ_ϵ. Since ϵ may be chosen as near zero as we wish, the k zeros of $f(z) + g(z)$ must be in or on the unit circle. Only one of these zeros ($z = 1$) is on the unit circle, so the other $k-1$ distinct zeros must be interior to the unit circle.

As noted above we may assume that the jth term of our k-generalized

Fibonacci sequence is expressible as a linear combination of the jth powers of the distinct roots $r_{1,k}, \cdots, r_{k,k}$ of $E_k'' = 0$, that is, that

$$f_{j,k} = \sum_{i=1}^{k} B_i (r_{i,k})^j, \qquad j = 0, 1, \cdots, k-1, k, \cdots. \tag{17}$$

Using (1) we see that the coefficients B_i are determined by the linear system

$$\begin{aligned} \sum_{i=1}^{k} B_i (r_{i,k})^m &= 0, \qquad m = 0, 1, \cdots, k-2, \\ \sum_{i=1}^{k} B_i (r_{i,k})^{k-1} &= 1, \end{aligned} \tag{18}$$

of k equations in k unknowns. In solving (18) by Cramer's rule we see that

$$B_i = N_i / D_k, \tag{19}$$

where D_k is the Vandermonde determinant

$$D_k = \begin{vmatrix} 1 & \cdots & 1 \\ r_{1,k} & \cdots & r_{k,k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_{1,k}^{k-1} & \cdots & r_{k,k}^{k-1} \end{vmatrix} = \prod (r_{m,k} - r_{n,k}), \tag{20}$$

where, in the product, $m > n$, $2 \leqq m \leqq k$, and N_i is obtained from D_k by replacing its ith column by a column of $k-1$ zeros followed by a 1. Expanding D_k by its ith column we see that it is $(-1)^{k+i}$ times a Vandermonde determinant of order $k-1$ involving all the $r_{n,k}$ except $r_{i,k}$. Thus we have

$$N_i = (-1)^{k+i} \prod (r_{m,k} - r_{n,k}), \tag{21}$$

where, in the product, $i \neq m > n \neq i$, $2 \leqq m \leqq k$. Substituting (21) and (20) in (19) we have

$$B_i = (-1)^{k+i} / \prod\nolimits_1 (r_{m,k} - r_{n,k}) = \prod\nolimits_2 (r_{i,k} - r_{n,k})^{-1}, \tag{22}$$

where, in $\prod_1$, $m = i$ or $n = i$, $m > n$, $2 \leqq m \leqq k$, and in $\prod_2$, $i \neq n$, $1 \leqq n \leqq k$. The last step in (22) follows on changing the sign of the $k-i$ factors $(r_{m,k} - r_{i,k})$, $m > i$, and introducing the compensating factor $(-1)^{k-i}$ to preserve equality. Thus we have

$$f_{j,k} = \sum_{i=1}^{k} \left[\prod\nolimits_2 (r_{i,k} - r_{n,k})^{-1} \right] (r_{i,k})^j. \tag{2}''$$

We note that (2)′ and (2) follow readily from (2)″ in the special case $k = 2$.

From (2)″ and the facts that (a) $|r_{i,k}| < 1$, $i < k$, and (b) $|r_{k,k}| > 1$, we conclude that $\lim_{j \to \infty} (f_{j,k} - B_k r_k^j) = 0$, whence we obtain

$$\lim_{j\to\infty} (f_{j+1,k})/(f_{j,k}) - r_{k,k} = 0, \tag{3}''$$

which includes (3)′ and (3) as special cases.

The matrix $A_{j,k}$ which occurs in the generalization we make of (4)′ is defined as follows. $A_{j,k}$ is the matrix with general element $a_{mn} = f_{j+m+n-2,k}$, $1 \leqq m, n \leqq k$; *i.e.*,

$$A_{j,k} = \begin{pmatrix} f_{j,k} & \cdots & f_{j+k-1,k} \\ f_{j+1,k} & \cdots & f_{j+k,k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_{j+k-1,k} & \cdots & f_{j+2k-2,k} \end{pmatrix}. \tag{7}''$$

We now show that

$$|A_{j,k}| = (-1)^{(2j+k)(k-1)/2}, \tag{4}''$$

which reduces to (4)′ or (4) for $k=2$.

We observe that the matrix $A_{0,k}$ is lower triangular with determinant

$$|A_{0,k}| = (-1)^{k-1}(-1)^{k-2}\cdots(-1)^1 = (-1)^{k(k-1)/2}. \tag{23}$$

We also note that

$$|A_{j,k}| = (-1)^{k-1}|A_{j-1,k}| \tag{24}$$

because $|A_{j-1,k}|$ is equivalent to the determinant $|A''_{j,k}|$ obtained from $|A_{j-1,k}|$ by replacing each element of its first column by its row sum, and $|A''_{j,k}|$ has identical columns with those of $|A_{j,k}|$ permuted cyclically so that the last column of $|A_{j,k}|$ is the first column of $|A''_{j,k}|$. Repeated application of (24) yields

$$|A_{j,k}| = (-1)^{j(k-1)}|A_{0,k}|. \tag{25}$$

By substitution of $A_{0,k}$ from (23) in (25) we may complete the proof of (4)″.

From (3)″ it follows that the rows of $A_{j,k}$ become almost dependent for large j. We obtain a crude measure of the ill-condition of these matrices from the following considerations.

The eigenvalues of the real symmetric matrix $A_{j,k}$ are the k real roots of an equation of form

$$\lambda^k - \sum_{n=0}^{k-1} f_{j+2n,k}\lambda^{k-1} + \cdots + (-1)^k(-1)^{(2j+k)(k-1)/2} = 0.$$

The largest eigenvalue of $A_{j,k}$ exceeds $f_{j,k}$ which is the smallest trace element. Thus for $j>k$ we have $\lambda_{k,k}>f_{j,k}>1$. Since the product of the eigenvalues has absolute value unity and one of them exceeds unity the smallest must be such that $|\lambda_{1,k}|<1$. This means that for fixed k and $j>k$ the P-condition number of $A_{j,k}$ exceeds $f_{j,k}$ which we have seen to be of the order of magnitude of $C(r_{k,k})^j$.

We consider a relatively unstable system of three equations associated with

the matrix $A_{3,3}$ under the assumption that all of its elements except a_{33} are exactly known. We may write the system in the following form.

$$4x + 7y + 13z = 24, \qquad 7x + 13y + 24z = 44, \qquad 13x + 24y + (44 + \epsilon)z = 81.$$

For certain values of ϵ corresponding to errors in a_{33} of less than 1% we note the following:

$$\begin{aligned} \epsilon &= 0: && (x, y, z) = (1, 1, 1), \\ \epsilon &= .3: && (x, y, z) = (-2, 13, 10), \\ \epsilon &= .4: && (x, y, z) = (3, 11, -5), \\ \epsilon &= 1/3: && \text{no solution.} \end{aligned}$$

We conclude with a generalization of the well-known fact that the ordinary Fibonacci numbers $f_{n,2}$ or f_n may be obtained by diagonal summing of the binomial coefficients arranged in a Pascal triangle. This familiar result, for which an inductive proof was given by Ganis [3], may be expressed analytically as follows:

$$f_n = \sum_{m=0}^{[(n-1)/2]} \binom{n-1-m}{m}. \tag{26}$$

In order to write the generalized form of (26) which the author has obtained, the following notation is introduced. The symbol $(a_1, \cdots, a_k)^*$ is defined to have the value

$$\left(\sum_{j=1}^{k} a_j\right)! \Big/ \prod_{j=1}^{k} (a_j!)$$

for a_j integral and nonnegative and the value zero otherwise (in particular if one of the a_j is negative). Consider ordered k-tuples $(a_1, \cdots, a_k)$; the set $S_{n,k}$ is defined as the set of all such k-tuples whose elements are nonnegative integers such that

$$\sum_{j=1}^{k} j \cdot a_j = n. \tag{27}$$

With the above notation (26) may be rewritten as

$$f_{n,2} = \sum_{S_{n-1,2}} (a_1, a_2)^*, \qquad n \geqq 1, \tag{28}$$

a special case of our general result

$$f_{n,k} = \sum_{S_{n-k+1,k}} (a_1, \cdots, a_k)^*, \qquad n \geqq k - 1, \tag{28}''$$

which we now establish. The proof depends upon the following extension to multinomial coefficients $(a_1, \cdots, a_k)^*$ of the familiar Pascal relation for bi-

nomial coefficients. We accept as our starting point this easily proved relation,

$$(29)\qquad (a_1, a_2, \cdots, a_k)^* = (a_1 - 1, a_2, \cdots, a_k)^* + (a_1, a_2 - 1, \cdots, a_k)^* + \cdots + (a_1, a_2, \cdots, a_k - 1)^*,$$

which holds for any set of k nonnegative integers $a_1, \cdots, a_k$, at least one of which is positive.

Now suppose that for a given k, and n replaced by each of the k consecutive integers $n-k, \cdots, n-1$, the relation (28)″ holds, we can show using (1) that (28)″ holds in general. Consider the result of replacing each term on the right hand side of (28)″ by its expansion in terms of (29). The replacement for a single term will consist of k terms, some of which may have the value zero, with the property that the jth such term belongs to the right-hand side of the assumed version of (28)″ in which $f_{n-j,k}$ is expressed as a sum over $S_{n-j-k+1,k}$. When the complete breakdown of the right-hand side of (28)″ for $f_{n,k}$ is accomplished in this fashion we note that the totality of terms on the right in this equation coincides with the totality of the corresponding terms in the expansions for $f_{n-1,k}, \cdots, f_{n-k,k}$. Furthermore these last quantities which occur on the left in our assumed expansions total $f_{n,k}$ by (1) and the desired result follows. There remains only the problem of demonstrating k consecutive values of n for which (28)″ holds. This may be observed to hold for $n = k-1, k, k+1, \cdots, 2k-2$ for which the values $f_{n,k}$ are successively $1, 1, 2, 4, \cdots, 2^{k-2}$. By the above arguments and the principle of induction, (28)″ holds for a fixed $k \geqq 2$ and all $n \geqq k-1$. We check the result for the case $n=9$ and $k=4$. The sequence $f_{n,4}$ goes $0, 0, 0, 1, 1, 2, 4, 8, 15, 29, \cdots$, so that $f_{9,4}=29$. On the other hand by (28)″ we get

$$\begin{aligned} f_{9,4} &= \sum_{S_{6,4}} (a_1, a_2, a_3, a_4)^* \\ &= (6,0,0,0)^* + (4,1,0,0)^* + (3,0,1,0)^* + (2,2,0,0)^* \\ &\quad + (2,0,0,1)^* + (1,1,1,0)^* + (0,3,0,0)^* + (0,1,0,1)^* + (0,0,2,0)^* \\ &= 1+5+4+6+3+6+1+2+1 = 29, \end{aligned}$$

which is what it should be.

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REMARKS ON A GENERALIZATION OF THE TRIGONOMETRIC FUNCTIONS

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Shelupsky [1] has recently investigated generalizations of the trigonometric functions defined by the differential equations

$$(d/dx)\alpha_s(x) = \beta_s^{s-1}(x), \qquad (d/dx)\beta_s(x) = -\alpha_s^{s-1}(x).$$

The purpose of this note is to point out prior work on this topic. Günther ([2], pp. 350–353) introduced these functions in terms of the geometry of the curves $x^n+y^n=1$ and defined, similarly, generalized hyperbolic functions in terms of the curves $x^n-y^n=1$.

The most exhaustive treatment appears to be that of Grammel [3]. He first defines the inverse circular functions

$$\text{ArCos}\,(n)x = \int_x^1 (1-t^n)^{(1-n)/n}dt,$$

$$\text{ArSin}\,(n)x = \int_0^x (1-t^n)^{(1-n)/n}dt$$

(the same as (5) and (6) of [1]), thence the direct functions, and proceeds to derive in considerable detail the properties of these functions, their series representations, and integral formulas. A similar treatment is given of the generalized hyperbolic functions. The results are extended to nonintegral values of n. Generalizations to curves defined by the more general relation $f(x, y)=1$ are also indicated. In another paper [4], Grammel has tabulated the values of the generalized circular functions for $n=4, 6, 8$.

It should be noted that the correct value for $\pi(3)$ is $3.533\cdots$, not $3.595\cdots$ as given in [1].

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3. R. Grammel, Eine Verallgemeinerung der Kreis- und Hyperbelfunktionen, Ing.-Arch., vol. 16, 1948, pp. 188–200. (A brief account under the same title appeared in Arch. Math., vol. 1, 1948, pp. 47–51.)

4. ———, Tafeln der verallgemeinerten Kreisfunktionen, Ing.-Arch., vol. 18, 1950, pp. 250–254.

ON THE "CLOCK PARADOX" IN RELATIVITY THEORY*

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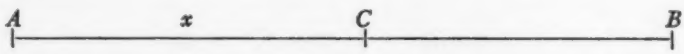
If of two newborn babies one were put on a space ship traveling with a speed v equal to 99% of the speed of light c to the star Alpha Centauri, which is about four light-years from the earth, then turned around and came back to the earth, he would be a little more than one year old while his twin brother would have grown to be about eight years old. The reason for this is that the boy on the space ship lives, grows, and ages according to his own time, the time of a coordinate system with respect to which he is at rest. In short one says: Clocks on moving coordinate systems are slower.

The "apparent" paradox arises from the question: What if we assume a frame of reference fixed to the space ship so that relative to this system the boy who stayed on earth is moving? Should not then the boy who stayed home turn out to be younger than his space-traveling twin?

We give the answer to this question in three parts.

1. To make sure we do not have to worry about the effect of the velocity-reversal of the space-ship after arrival at Alpha Centauri we make the following arrangement:



We reflect the distance x from earth (A) to Alpha Centauri (C) on C which leads to B with $\overline{AC}=\overline{BC}$. Assume a colony of astronauts at B who are able to send newborn babies towards C. Have an astronaut at C release a light-flash at a certain moment. At the moment of the arrival of this light-flash a baby is sent out from earth (A) and from the space station (B) towards C. We first concentrate our attention on the earth baby traveling from A to C; when he meets the other baby at C we switch our attention to this other baby traveling from C to A. From the symmetry of this arrangement it is clear that the time necessary for the round trip $A\rightarrow C\rightarrow A$ is equal to the time required for the trip from A to B, or twice the time necessary for traveling from A to C.

2. We determine this time necessary to travel from A to C in both coordinate systems.

In the system resting with our solar system, the space ship traveling with velocity v obviously needs the time $t=x/v$ to cover the distance x according to

$$x = vt. \tag{1}$$

In the language of relativity theory, x and t are the world coordinates of the event (x, t), which is the arrival of the space ship at C. In the coordinate system in which the space ship is at rest, this same event of the ship's arrival at C has

* Cf. Alfred Schild, The clock paradox in relativity theory, this MONTHLY, vol. 66, 1959, pp. 1–18.

coordinates (x', t'). These coordinates are related to x and t by the Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{\{1 - (v^2/c^2)\}}}; \qquad t' = \frac{t - (v/c^2)x}{\sqrt{\{1 - (v^2/c^2)\}}}. \tag{2}$$

Substituting from (1) into (2) leads to

$$x' = 0; \qquad t' = t\sqrt{\{1 - (v^2/c^2)\}}. \tag{3}$$

The first of equations (3) confirms what we already know: the arrival of the ship at C takes place at a distance zero from the ship and nowhere else. The second of equations (3) tells us that, according to ship's time, this event will occur at an earlier time. If, according to a terrestrial observer, the ship needs $t=4$ years to cover the distance of four light-years from earth to Alpha Centauri, then the ship's clocks will read $t' = 4\sqrt{\{1-(99/100)^2\}}$ or approximately half a year.

3. Now for the "paradox." Taking the coordinate system that moves along with the ship as the primary system, we should say: At the beginning of the voyage from A to C the earth was at the origin of this coordinate system $(x'=0)$; during the voyage, it moves backward with velocity $-v$ until C is at the origin and A is a distance $-x'$ behind the ship and

$$-x' = -vt'. \tag{4}$$

Since A and C are receding with velocity $-v$ relative to the space ship, this distance $\overline{AC}$ appears Lorentz-contracted for physicists on the space ship. Therefore

$$x' = x\sqrt{\{1 - (v^2/c^2)\}}. \tag{5}$$

Because of the Lorentz contraction (5), equation (4) becomes $-x\sqrt{\{1-(v^2/c^2)\}} = -vt'$ or $t' = (x/v)\sqrt{\{1-(v^2/c^2)\}}$. Since from (1) we know that $x/v = t$, we again have the second of equations (3). Thus, no matter which coordinate system is used, the twin who did the traveling is always younger by a factor $\sqrt{\{1-(v^2/c^2)\}}$.

MATHEMATICAL COMPETITIONS IN CHINA*

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Widespread American interest in the state of mathematics education in the Soviet Union was not aroused until the launching of the first Sputnik. Lacking any comparable incentive there has as yet been little concern with this subject as it applies to China. Yet it would seem advisable to make some beginning now in looking into this question, as there are several reasons for expecting a significant increase in the number and quality of mathematicians in China.†

China's population of some half to two-thirds of a billion people provides a vast reservoir of potential talent. Throughout its history this populous society has placed great store on intellectual activity, as is attested, among other things, by the fact that the Chinese equivalent of our Horatio Alger was the poor but bright young man who rose to high position by success in the rigorous competitive examinations for the imperial civil service. Today under the new regime on the mainland the rewards for academic success, both in terms of prestige and material advantage, are perhaps greater than ever before. This is particularly true in such officially preferred areas as mathematics.

One of the ways in which this official preference manifests itself is in the promotion of mathematical competitions as a means of identifying and encouraging mathematically talented students. Such contests were held for the first time in 1956 in four major cities in China. The Chinese Mathematical Society played a leading role in the actual conduct of the competitions. As a guide in the anticipated expansion of this activity throughout the country the Middle School Mathematics Study Group of the Shanghai Branch of the Chinese Mathematical Society published a report‡ on the competitions in China's largest city. I present herewith some notes based on this report together with a full translation of the contest questions used in Shanghai.

In broad perspective the mathematical competitions were undertaken in

* I am indebted to the American Council of Learned Societies and the Social Science Research Council for a joint grant in support of a project on Chinese mathematics of which this study is a part.

† The use of the phrase "mathematicians in China" rather than "Chinese mathematicians" is intended to exclude from consideration those of the latter who are even now serving with distinction in our own institutions of higher learning.

‡ Shang-hai shih 1956–57 nien chung hsüeh-sheng shu-hsüeh ching-sai his-t'i pien-hui [Compilation of problems from the 1956–57 Mathematical Competitions for Middle-School Students in Shanghai Municipality], Shanghai, Hsin chih-shih, 1958.

response to the official summons to the Chinese people to catch up with the advanced industrial nations of the world by 1970. (Great Britain is the rival most frequently mentioned in connection with this goal.) The report stresses the need for a rapid rise in the level of mathematical competence so as to satisfy the demands of the "great leap forward," particularly in such areas as agriculture, engineering, and the natural sciences.

The four major cities of Shanghai, Peking, Tientsin, and Hankow were the first to undertake mathematical competitions in 1956, their example being followed by many more cities in the following year. It was expected that eventually the contests would spread so widely that they would be comparable in scope to the Mathematical Olympiads of the Soviet Union and other countries of Eastern Europe.§

As to the contests in Shanghai, at the beginning of 1956 the local branch of the Chinese Mathematical Society, acting at the behest of the parent organization, proceeded to organize a Committee on Mathematical Competitions for Middle School Students. The 17 members of the committee were drawn from the Mathematical Society, the Shanghai Office of Education, and the Shanghai Branch of the All-China United Association of Natural Science Professional Societies. Support was also obtained from teachers and various social organizations. In addition the press and radio lent a hand in providing city-wide publicity to supplement the efforts made directly to the students in school to arouse interest in the contests.

The Shanghai competitions of 1956 and 1957 were held at two levels, one for second-year and another for third-year upper middle school students. (The Chinese lower- and upper-middle schools are of three years each and include grades 7 through 12. Hence the contestants corresponded approximately to our high school juniors and seniors.) Each competition consisted of three rounds. The first or preliminary round was conducted by the schools with their own examinations. Of the students enrolled in the last year of upper middle school, more than 4000 participated in the preliminary competition in 1956, and many more did so in 1957, but this was still less than 50 per cent of the total. The schools selected the best students to participate in the second round of the contest, the number chosen being limited to 3 per cent of the students enrolled in each class. There were 732 second-round contestants in 1956, and 1063 in 1957. The second and third rounds of the competitions were conducted by the Committee on Mathematical Competitions, which selected the top 6 to 10 per cent of the second-round contestants for participation in the third round, and then designated 10 students from each class as winners of the final round.

At the conclusion of each competition an awards assembly was held so as to increase the enthusiasm of the students who had already participated and to

§ R. Creighton Buck, A look at mathematical competitions, this MONTHLY, vol. 66, 1959, pp. 201–212; J. Aczél, A look at mathematical competitions in Hungary, this MONTHLY, vol. 67, 1960, pp. 435–437; Izaak Wirszup, The seventh mathematical olympiad for secondary-school students in Poland, The Mathematics Teacher, vol. 51, 1958, pp. 585–589.

encourage other students to participate in the future. Those attending the assembly included winners in each of the three rounds, student representatives of the first year of upper middle school, teachers, principals, parents of the third-round winners, and representatives of social organizations. The best students in the second- and third-round competitions were given certificates of merit and awards. In addition a representative of the Central Ministry of Higher Education made the announcement that the three top students in the final competition for the third-year class would be permitted to study any speciality in mathematics, astronomy, physics, and allied fields in any specified local comprehensive university without being subjected to further examination.*

There was one reservation as to the qualifications of these winners. This was that the student had to be in good health and politically acceptable. The feeling was expressed that inadequate attention had been given to this last point and that it should be taken into consideration in the future in selecting contestants for the second and third rounds. Ideally the end product of the educational process was a person who was, as the slogan put it, "both a Communist and a specialist."

As to the specifically mathematical aspects of the competitions, both positive and negative results were reported. On the positive side was the fact that the top ten names in the final round of the 1957 competition for third-year middle school students were distributed among nine schools, and the top 44 names in the second round were distributed among 26 schools, a picture which was said to be true also of the competitions for second-year students. This was considered evidence of the high quality of mathematics teaching throughout the middle schools of Shanghai. Nevertheless the judgment was expressed that the level of mathematical competence among students was still quite inadequate. Only fragmentary data were reported on student performance on the individual test problems, but some specific comments were made with respect to general shortcomings. These were as follows:

1. Many students are deficient in analyzing a problem on the basis of hypothesis and conclusions, getting at the heart of the problem, and then applying logical reasoning to its solution. Students should be given more opportunity for independent analysis and synthesis. It would be well for them to think about some problems for several days and not to be given the solutions too quickly.

2. Students should be given more practice in handling problems which involve a synthesis of algebra, geometry, and trigonometry.

3. More attention should be devoted in middle school to the study of inequalities so as to facilitate the transition from middle school to college mathematics.

4. Students pay insufficient attention to restating problems and are inadequately skilled in performing operations. The latter was especially apparent in complicated problems. Hence students should be given comparatively complicated problems for homework so as to develop their operational skills.

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* Although students attending institutions of higher learning do not pay tuition fees and are entitled to receive grants in aid which provide for basic living expenses, the number of students attending such institutions is limited by competitive examinations.

It was further recommended that in order to enable students to go beyond the basic concepts and theorems contained in their textbooks, they should be encouraged to read additional books and journals, to form study groups for discussion of problem-solving and extracurricular reading, and to participate in regular, small-scale mathematical competitions.*

The emphasis in the preceding remarks on the importance of mathematical reasoning is reflected in the choice of problems for the competitions. Although these undoubtedly call for factual knowledge, they are primarily of the perceptive or aptitude type. Only five or six problems were assigned in any one competition, and 150 minutes, or about half an hour per problem, was allotted for their solution.

The problems for the second and third rounds of the 1956 and 1957 competitions are given below.

Problems of the second round, 1956

1. (a) Given a positive integral number n, prove that $13^{3n}-1$ is a multiple of 168.

(b) What is the nature of the natural number n that will make $1\cdot 2\cdot 3\cdot \cdots \cdot n$ divisible by $1+2+3+\cdots+n$?

2. Solve the inequality $10^{\log^2 x}+4x-\log_2 32>0$.

3. Between two low-lying fields is an angled ditch (Fig. 1). It is desired to straighten both banks of the ditch without changing the area of either field.

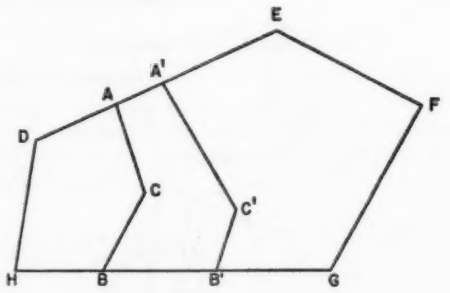

FIG. 1

(a) If the points A, A' are not changed, what should be done to the ditch?

(b) If the point A is not changed and the two banks of the ditch are to be parallel, what should be done to the ditch?

Explain the method of procedure in each case.

4. Find the six-digit number $1abcde$ which when multiplied by 3 becomes $abcde1$.

5. Construct a circle and, using only compasses (not a straight edge), divide the circle into 4 equal parts.

6. From a point A due west of an air-borne balloon the angle of elevation is 45°. From a point

* An interesting application of this emphasis on collective work and mathematical competitions is the combination of the two in the form of mathematical relay races. These are described by a middle school teacher, Ting Han-wen, in an article entitled "Shu-hsüeh chieh-li sai [Mathematical Relay Races]", Chung-hsüeh shu hsüeh [Middle School Mathematics], No. 7, 1959, 19–20.

B due southeast the angle of elevation is 67°30′. The points A and B are 266 meters apart and are both one meter above the ground. How high above the ground is the balloon?

Problems of the final round, 1956

1. From among the natural numbers from 1 to 100, in how many ways is it possible to select two numbers such that their sum exceeds 100?

2. (a) Given the arithmetic progression $a_1, a_2, a_3, \cdots, a_{n+1}$, prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n+1}}}=\frac{n}{\sqrt{a_1}+\sqrt{a_{n+1}}}.$$

(b) Given the geometric progression $a_1, a_2, a_3, \cdots, a_{n+1}$, prove that

$$\sqrt{(a_1a_2)}+\sqrt{(a_2a_3)}+\cdots+\sqrt{(a_na_{n+1})}=\frac{\sqrt[n]{(a_1a_{n+1})}}{\sqrt[n]{a_1}-\sqrt[n]{a_{n+1}}}(a_1-a_{n+1}).$$

3. Given the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ with integral coefficients, and an odd number α and an even number β such that $f(\alpha)$ and $f(\beta)$ are both odd. Prove that the equation $f(x)=0$ has no integral roots.

4. Given any $\triangle ABC$, prove that

(a) $tg^2\frac{1}{2}A+tg^2\frac{1}{2}B+tg^2\frac{1}{2}C\geqq 1$; (b) $\cos A+\cos B+\cos C>1$.

5. In a given circle construct an inscribed equilateral triangle such that the sum of its base and its height is a maximum.

6. Figure 2 represents a flattened solid. That is to say, if properly put together it will form a solid.

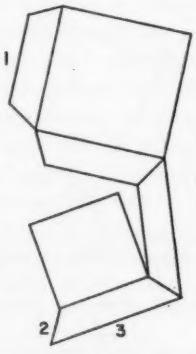

FIG. 2

(a) When the figure is assembled, which line segments meet line segments 1, 2, 3 respectively? Write in the corresponding numbers.
(b) What is this figure? Why?

Problems of the second round, 1957

A. Problems for second-year students

1. Find 10 consecutive natural numbers each of which is a composite number.

2. If the equations $x^2+ax+b=0$ and $x^2+px+q=0$ have one root in common, find the quadratic equation whose roots are their dissimilar roots.

3. For what values of x is the following meaningful:

$$\sqrt{\lg\sqrt{\lg\sqrt{\lg\sqrt{\lg\sqrt{\lg\sqrt{\lg x}}}}}}.$$

4. About a given equilateral triangle circumscribe another equilateral triangle whose area is a maximum.

5. If A is an acute angle, prove that

$$\begin{aligned}&\sec A + \sec(A/2) + \sec(A/3) + \cdots + \sec(A/n)\\ &+ \csc A + \csc(A/2) + \csc(A/3) + \cdots + \csc(A/n)\\ &> \sec A \csc A + \sec(A/2)\csc(A/2) + \sec(A/3)\csc(A/3) + \cdots + \sec(A/n)\csc(A/n).\end{aligned}$$

B. Problems for third-year students

1. Same as Problem 1 for second-year students.

2. Find the positive root of the following equation, and prove that there is only one positive root:

$$x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+x}}}}.$$

3. Find the real values of x which will satisfy the following relationship: $|3x-2|+|3x+1|=3$.

4. D, C are two points on a semicircle with diameter AB, and x is any point on AB. Prove that $\operatorname{tg} ACX \cdot \operatorname{tg} BDX = \operatorname{tg} BAC \cdot \operatorname{tg} ABD$.

5. $V-ABC$ is a trihedral angle and VD is a line bisecting the face angle BVC. If $\angle AVD \gtreqless \frac{1}{3}\pi$, prove that $\frac{1}{2}(\angle AVB+\angle AVC) \lesseqgtr \angle AVD$.

6. Prove that in any triangle the continued product of the lines bisecting the three angles is less than the continued product of the three sides.

Problems of the final round, 1957

A. Problems for second-year students

1. If the natural number $62ab427$ is a multiple of 99, find a, b.

2. Given a rectangular plot of land $ABCD$ which is divided into three segments by the straight lines EF and GH. (Fig. 3). How can these three segments be transformed into rectangles of respectively equal area such as in Figure 4?

FIG. 3

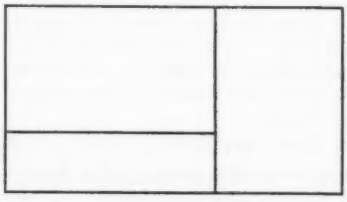

FIG. 4

3. If $x+y+z=0$, prove that

$$\left(\frac{x^3+y^3+z^3}{2}\right)\left(\frac{x^5+y^5+z^5}{5}\right) = \frac{x^7+y^7+z^7}{7}.$$

4. Given four concentric circles whose radii are respectively 1, 2, 3, 4 and the square $ABCD$ circumscribed about the largest circle.

(a) Make a square contained in $ABCD$ whose four sides are respectively tangent to the four circles* and are parallel to or identical with the four sides of $ABCD$.

(b) How many such squares can be made?

5. (a) In any given triangle ABC with sides a, b, c (Fig. 5), prove that $a^2=b^2+c^2-2bc\cos A$.

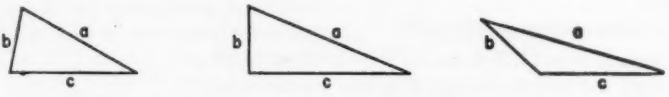

FIG. 5

(b) Given a point P on the circumference of a unit circle and the vertices $A_1, A_2, \cdots, A_n$ of an inscribed regular polygon of n sides (Fig. 6), prove that $\overline{PA_1}^2+\overline{PA_2}^2+\cdots+\overline{PA_n}^2$ is a constant.

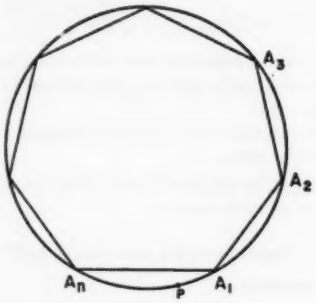

FIG. 6

B. Problems for third-year students

1. Same as Problem 1 for second-year students.

2. Same as Problem 2 for second-year students.

3. Given the whole number a larger than 2 and the composite number b ($b>0$), if b can be divided evenly by r different positive integral numbers, prove that a^b-1 can be divided evenly by at least r different positive integral numbers.

4. Prove that the area of any polygon of n sides inscribed in a unit circle does not exceed $\frac{1}{2}n\sin(2\pi/n)$.

5. Given the points $A_1, A_2, A_3, \cdots$, on a straight line, with $\overline{A_1A_2}$ having length 1; A_{n+2} a point on the segment $\overline{A_nA_{n+1}}$; and $\overline{A_nA_{n+2}}$: $\overline{A_{n+2}A_{n+1}}=\overline{A_{n+2}A_{n+1}}$: $\overline{A_{n+1}A_n}$, in which $n=1, 2, 3, \cdots$. Prove that the sequence formed by the lengths $\overline{A_1A_2}$, $\overline{A_1A_3}$, $\overline{A_1A_4}$ has a limit, and that this limit is $\frac{1}{2}(\sqrt{5}-1)$ (Fig. 7).

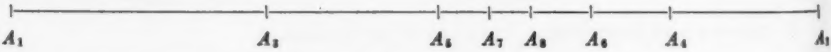

FIG. 7

* It is not clear from the wording of the problem that in some cases the sides of the square must be extended if they are actually to touch the circles.

ON GENERALIZATIONS OF THE EXPONENTIAL FUNCTION

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1. Introduction. This paper deals with functions that are defined as infinite series of the nth power (n fixed) of the terms of the Maclaurin series of the exponential function. Such functions are primarily met in the study of generating functions of sets of orthogonal functions. To our knowledge such functions have neither been studied nor tabulated although they seem useful (aside from the above-mentioned application) as a generalization of the exponential function and contain, for their second order, a certain kind of Bessel functions.

It will be shown that there is a linear relation between the function of order n and its derivatives up to order n; thus the function satisfies a linear differential equation of order n. The n linearly independent solutions of the differential equation are found as logarithmic series. It follows that the function can be expanded in logarithmic series.

2. Notations. The following notations are currently used in finite differences (see, for example, [1]), a being any complex number and b any positive integer.

Factorial monomial:

$$(a)_b = a(a-1)\cdots(a-b+1). \tag{1}$$

Stirling numbers of the first kind: St_b^m $(1 \leqq m \leqq b)$ defined by

$$(a)_b = \sum_{m=1}^{b} St_b^m a^m. \tag{2}$$

Stirling numbers of the second kind: st_b^m $(1 \leqq m \leqq b)$ defined by

$$a^b = \sum_{m=1}^{b} st_b^m (a)_m. \tag{3}$$

Differential operators:

$$D_t = d/dt, \qquad D_x = d/dx, \qquad D_s = d/ds. \tag{4}$$

3. Definition of the function T_n. Differential equation. Let n be a positive integer; we define the function T_n by:

$$T_n(x) = \sum_{m=0}^{\infty} x^{mn}/(m!)^n. \tag{1}$$

The general term of the series is the nth power of the general term of the exponential series. The series converges, just as the exponential series does, for all values of the variable. We denote by T_n^p the pth derivative of T_n. Hence for any positive integer $p \leqq n$,

$$T_n^p(x) = D^p T_n(x) = \sum_{m=1}^{\infty} (nm)_p x^{nm-p}/(m!)^n. \tag{2}$$

If $n<p$, the expressions are the same, but the summation will start with 2 for $n<p\leqq 2n$; with 3 for $2n<2p\leqq 3n$; etc. For the special case $p=n$, we have

$$D_x^n T_n(x) = T_n^n(x) = \sum_{m=1}^{\infty} (nm)_n x^{nm-n}/(m!)^n. \tag{3}$$

T_n^n can be expressed in terms of T_n^{n-1}, T_n^{n-2}, $\cdots$, T_n^1, and T_n with the help of Stirling numbers of the first and second kind. Direct calculations are long and complicated. It is simpler to make the substitution $x=e^{t/n}$, $t=n \ln x$. We thus obtain

$$T_n(e^{t/n}) = \sum_{m=0}^{\infty} e^{mt}/(m!)^n, \tag{4}$$

$$D_t^n T_n(e^{t/n}) = e^t T_n(e^{t/n}), \tag{5}$$

so that

$$(D_t^n - e^t) T_n(e^{t/n}) = 0, \tag{6}$$

which proves that $T_n(e^{t/n})$ is a particular solution of the differential equation

$$(D_t^n - e^t)y = 0. \tag{7}$$

It is then easy to return to x, since $dx/dt=x/n$, and $D_t^2 y=(xD_x y+x^2D_x^2 y)/n^2$. More generally it is found that

$$D_t^p y = n^{-p} \sum_{q=1}^{p} A_p^q x^q D_x^q y, \tag{8}$$

where the A_p^q are numerical constants.

It is easily seen that the A_p^q-coefficients satisfy the relation $A_{p+1}^q = A_p^{q-1} + qA_p^q$ if we define $A_p^0 = A_p^{p+1} = 0$. This is the same relation as the one satisfied by Stirling numbers of the second kind (cf. [1], p. 169). Since $A_1^1 = A_2^1 = A_2^2 = 1$, it follows that the A_p^q-coefficients are no other than the Stirling numbers of the second kind, $A_p^q = st_p^q$, and (8) becomes for $p=n$

$$D_t^n y = \sum_{q=1}^{n} st_n^q x^q D_x^q y/n^n. \tag{8a}$$

Then (7) becomes

$$\left[\sum_{q=1}^{n} st_n^q x^{q-1} D_x^q - n^n x^{n-1} \right] y = 0. \tag{9}$$

Equation (9) has the following special cases:

$$y' - y = 0 \qquad (n = 1),$$

$$xy'' + y' - 4xy = 0 \qquad (n = 2),$$

$$x^2y''' + 3xy'' + y' - 27x^2y = 0 \qquad (n = 3),$$

$$x^3y''' + 6x^2y''' + 7xy'' + y' - 256x^3y = 0 \qquad (n = 4).$$

For $n=1$ the solution is $y=Ce^x$. For $n=2$ the solution is given in [2], p. 440; there equation 2.162 (1a) is $x^2y''+axy'+(bx^m+c)y=0$, which for $a=1$, $b=-4$, $c=0$, $m=2$, has the solution $y=C_1J_0(2ix)+C_2Y_0(2ix)=C_1I_0(2x)+C_2K_0(2x)$. For a tabulation of Stirling numbers of the second kind see [1], p. 170.

From (9) we obtain $T_n^n=n^nx^{n-1}-\sum_{q=1}^{n-1} st_n^q x^{q-1}T_n^q$ which expresses T_n^n in terms of T_n^k, $k<n$, and Stirling numbers. For example, we have

$$T_1^1 = T_1, \qquad T_2^2 = -(T_2^1 - 4xT_2)/x, \qquad T_3^3 = -(3xT_3^2 + T_3^1 - 27x^2T_3)/x^2,$$

$$T_4^4 = -(6x^2T_4^3 + 7xT_4^2 + T_4^1 - 256x^3T_4)/x^3.$$

4. General solution of the differential equation. Equation (9), Section 3, is of Fuchsian type (Cf., [4], vol. II). The point $x=0$ is a regular singular point of the differential equation. We obtain the indicial equation by looking for solutions of the form

$$y = \sum_{m=0}^{\infty} a_m x^{c+m}, \qquad a_0 \neq 0. \tag{1}$$

Since $D_x^q y=\sum_{m=0}^{\infty} a_m(c+m)_q x^{c+m-q}$, we obtain by substituting into (9), Section 3,

$$\sum_{q=1}^{n}\sum_{m=0}^{\infty} a_m(c+m)_q st_n^q x^{c+m-n} - n^n \sum_{m=0}^{\infty} a_m x^{c+m} = 0.$$

Equating to zero the coefficient of x^{c+m} and using (3), Section 2, we obtain $a_{m+n}(c+m+n)^n-n^na_m=0$ or $a_m(c+m)^n-n^na_{m-n}=0$, if we let $a_j=0$ for $j<0$. For $m=0$ this last expression reduces to $a_0c^n=0$, and since $a_0\neq 0$, this equation has n roots in c equal to zero. According to Fuchs' theory ([4] vol. II) it follows that (9), Section 3, has one single particular solution in power series, while all the other particular solutions contain logarithmic terms or can be expressed in power series of logarithms. This last form is easily obtained from (7), Section 3 by searching for solutions of the form $y=\sum_{m=0}^{\infty} a_m t^{c+m}$ to the differential equation $(D_t^n-e^t)y=0$. By direct substitution we obtain, after division by t^c,

$$\sum_{m=0}^{\infty} a_m(m)_n t^{m-n} - \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_q t^{p+q}/p! = 0.$$

Taking $a_0, a_1, \cdots, a_{n-1}$, as indeterminate coefficients, we obtain by equating equal powers of t, for $m>n-1$,

$$(n+p)_n a_{n+p} = \sum_{q=0}^{p} a_q/(p-q)! \qquad p = 0, 1, 2, \cdots.$$

The coefficients $a_0, a_1, \cdots, a_{n-1}$ can be chosen arbitrarily and the remaining coefficients a_{n+p} determined from them. In particular, we can obtain n independent solutions

$$y_{q,n}(t) = \sum_{m=0}^{\infty} a_m^q t^m, \qquad q = 0, 1, \cdots, n-1$$

by taking

$$a_m^q = \begin{cases} 1 \text{ if } m = q \\ 0 \text{ if } m \neq q, \end{cases} \qquad 0 \leqq m \leqq n-1.$$

It is convenient to write $N(n, q, m) = a_m^q m!$. The following table gives the values of $N(n, q, m)$ for $n = 2, 3, 4$, $m \leqq 10$:

m	0	1	2	3	4	5	6	7	8	9	10
$N(2, 0, m)$	1	0	1	1	2	5	13	36	109	359	1266
$N(2, 1, m)$	0	1	0	1	2	4	10	29	90	295	1030
$N(3, 0, m)$	1	0	0	1	1	1	2	6	17	54	98
$N(3, 1, m)$	0	1	0	0	1	2	3	5	12	48	89
$N(3, 2, m)$	0	0	1	0	0	2	6	12	22	48	96
$N(4, 0, m)$	1	0	0	0	1	1	1	1	2	7	23
$N(4, 1, m)$	0	1	0	0	0	1	2	3	4	6	14
$N(4, 2, m)$	0	0	1	0	0	0	2	6	12	20	32
$N(4, 3, m)$	0	0	0	1	0	0	0	6	24	60	120

The particular solutions of (7), Section 3, can now be written

$$y_{q,n}(t) = \sum_{m=0}^{\infty} N(n, q, m) t^m/m!, \qquad q = 0, 1, \cdots, n-1, \tag{2}$$

and the solutions of (9), Section 3, as

$$y_{q,n}(n \ln x) = \sum_{m=0}^{\infty} N(n, q, m) n^m (\ln x)^m/m!. \tag{3}$$

The solution $T_n(x)$ is a linear combination of the $y_{q,n}(x)$,

$$T_n(x) = \sum_{q=0}^{n-1} K_q y_{q,n}(x) = \sum_{m=0}^{\infty} \sum_{q=0}^{n-1} K_q N(n, q, m) n^m (\ln x)^m/m!.$$

Thus we have expressed $T_n(x)$ as a power series in $\ln x$. In the case $n = 2$, for example, we obtain

$$T_2(e^{t/2}) = \sum_{m=0}^{\infty} e^{mt}/(m!)^2 = K_0 y_{0,2}(t) + K_1 y_{1,2}(t),$$

$$D_t T_2(e^{t/2}) = \sum_{m=0}^{\infty} me^{mt}/(m!)^2 = K_0 y'_{0,2}(t) + K_1 y'_{1,2}(t).$$

Taking $t=0$, we obtain $y_{0,2}(0)=1$ and $y_{1,2}(0)=0$, $y'_{0,2}(0)=0$, and $y'_{1,2}(0)=1$, so that

$$K_0 = T_2(1) = 2.279\ 585\ 518\cdots, K_1 = \sum_{m=1}^{\infty} m/(m!)^2 = 1.590\ 638\cdots.$$

For other values of n the calculations are similar.

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MATHEMATICAL NOTES

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Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE TWIN PRIME CONSTANT

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Let $T(x)$ be the number of prime twins with first member not exceeding x. Brun [1] showed that $T(x)=O(x/\log^2 x)$, and Hardy and Littlewood [2] proved that if $T(x)\sim C(x/\log^2 x)$ for some constant C (which is a plausible conjecture with excellent empirical support from the tables of prime numbers) then

$$C = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) = 1.32032\cdots. \tag{1}$$

In 1954, E. M. Wright [3], in his *Postscript on prime pairs*, attributed a non-rigorous probabilistic derivation of (1) to Lord Cherwell, citing a 1946 paper [4]. This reference was repeated recently in this MONTHLY [5] by Pólya. Actually, the first probabilistic derivation of (1) can be found explicitly as early as Selmer's paper [6] which appeared in 1942, and perhaps implicitly in earlier writings of Brun (cf. [7]).

The probabilistic derivation of (1) is actually simpler than the presentations of it in [3], [5], and [6]. It may be done in this manner:

By the prime number theorem, Pr(x is prime)$\sim 1/\log x$. However, the proba-

bility that $x+2$ is prime, *given* that x is prime, departs from $1/\log(x+2)$ as follows: Since x is prime, x is odd with probability 1, and $x+2$ is odd, doubling the chance of $x+2$ being prime. For any odd prime p, $x \neq p$ is *not* divisible by p with probability 1, which *decreases* the probability that $x+2$ is relatively prime to p from $(p-1)/p$ to $(p-2)/(p-1)$. Thus the probability that $x+2$ is prime departs from $1/\log(x+2)$ by the factor

$$2\prod_{p>2}\left(\frac{p-2}{p-1}\right)\Big/\left(\frac{p-1}{p}\right) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

Thus

$$T(x)/x \sim \frac{1}{\log^2 x}\cdot 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

The question of how many primes to include in the product need never arise (as it does in the references cited), because the infinite product is clearly convergent. Thus the product can be extended over all odd primes, rather than starting with $2<p<x^{\mu}$ and letting $x\to\infty$, where x^{μ} can only be justified on seemingly mystical grounds [5] at best. (In fact, different authors use different values of μ.)

In fairness to the references cited, the question of how many primes to include in the product *is* significant in estimating $\pi(x)$, the number of primes up to x, but as shown in this note (which follows the treatment in [8]), the question can be sidestepped entirely in the probabilistic estimation of $T(x)$.

It is also interesting to observe that

$$C = 2\sum_{\text{odd } n=1}^{\infty}\frac{\mu(n)}{\phi^2(n)}, \tag{2}$$

where μ and ϕ are the Möbius and Euler functions, respectively. This formula arises in the "singular series" approach of Hardy and Littlewood. Actually, (1) may be regarded as an Euler product expansion for (2).

Also,

$$C = \frac{1}{4}\sum_{\text{odd } n=1}^{\infty}\frac{\mu(n)2^{\nu(n)}\log^2 n}{n}, \tag{3}$$

where $\nu(n)$ is the number of distinct prime factors of n. The expression (3) for C arises in the approach to twin primes followed in [8].

Another interesting constant connected with the twin primes is "Brun's sum,"

$$(\tfrac{1}{3}+\tfrac{1}{5})+(\tfrac{1}{5}+\tfrac{1}{7})+(\tfrac{1}{11}+\tfrac{1}{13})+(\tfrac{1}{17}+\tfrac{1}{19})+\cdots,$$

the sum of the reciprocals of the twin primes, which Brun showed is either convergent or finite. Selmer [6] evaluated this sum to three decimal places,

omitting the pair $(1/3+1/5)$, and obtaining the value $1.368\cdots$ for the rest, guaranteed to within 0.1 per cent.

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THE QUADRATIC SUBFIELD OF THE FIELD GENERATED BY THE p-th ROOT OF UNITY

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It is well known that the field $R(\zeta_p)$ generated by the pth root of unity ζ_p (p an odd prime) contains the field generated by $\sqrt{p}$ as its only quadratic subfield if $p\equiv 1\ (4)$ and the field generated by $\sqrt{(-p)}$ if $p\equiv 3(4)$. An elementary proof for this discrepancy follows from the Gauss sums. Another proof can be obtained from the known expression for the discriminant of the cyclotomic field which contains p as its only prime factor. This implies that the quadratic subfield must be generated by the square root of an integer $n\equiv 1(4)$. The following alternative proof uses no algebraic number theory.

Let $p=4n+3$. Then $p-1=4n+2=2m$ where m is odd. Since $(2, m)=1$ the field $R(\zeta_p)$, which is cyclic and of degree $p-1$, is the product of a quadratic field and of a field of odd degree which is necessarily real. Since $R(\zeta_p)$ contains complex quantities the quadratic field cannot be real.

For $p=4n+1$ we can argue, *e.g.*, as follows: Since $p-1=4n$ the field $R(\zeta_p)$ contains a biquadratic cyclic field and the quadratic subfield of such a field is known to be real.*

In a vague way, a transition between the two cases is given by the case of the integers of the form $4n$. The field $R(\zeta_{4n})$, $n>1$, is no longer cyclic and contains a real and an (actually several) imaginary quadratic field. For, the field of the 8th root of unity contains $\sqrt{(-1)}$ and $\sqrt{2}$, while any $R(\zeta_{4p})$, p an odd prime, contains $\sqrt{(-1)}$ and $\sqrt{p}$ or $\sqrt{(-p)}$, hence both.

* See, *e.g.*, H. Weber, Algebra, vol. 1 (2nd ed.), 1898, p. 699.

THE BASIS THEOREM FOR FINITELY GENERATED ABELIAN GROUPS

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The following proof of the basis theorem for finitely generated Abelian groups may be of interest because of its simplicity and because of the theorem presented here on which it is based.

If a_1 and a_2 are integers with greatest common divisor 1 then there are numbers r_1 and r_2 so that $a_1r_1+a_2r_2=1$. This fact is extended in the lemma below.

LEMMA. *If $a_1, \cdots, a_n (n>1)$ are integers whose greatest common divisor is* 1, *then there is an $n\times n$ matrix with integral elements whose determinant is* 1 *in which the $a_1, \cdots, a_n$ appear as the elements of the first row.*

Proof. For $n=2$ this follows from the statement of the paragraph preceding the lemma. We suppose that for $i=2, \cdots, n-1$, $a_i=b_id$ where d is the greatest common divisor of the a_i and hence the b_i have greatest common divisor 1; then by the induction assumption the b_i are elements of the first row of an $n-1$ square matrix of determinant 1.

Since a_n and d have greatest common divisor 1, there are integers s and t so that $sa_n+dt=1$. If e is the proper choice of ± 1 and if the $n-1$ square matrix of the previous paragraph is bordered as follows:

$$\begin{pmatrix} b_1d & \cdots & b_{n-1}d & a_n \\ * & \cdots & * & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ * & \cdots & * & 0 \\ esb_1 & \cdots & esb_{n-1} & t \end{pmatrix},$$

the resulting matrix has determinant 1 and the lemma is proved.

THEOREM. *If $x_1, \cdots, x_n$ are generators of an additively written Abelian group and $a_1, \cdots, a_n$ are integers with greatest common divisor* 1, *then $a_1x_1+\cdots+a_nx_n$ may be chosen as one of a set of n generators of the group.*

Proof. Let A be the matrix of the lemma. Then $A^{-1}=$Adjoint of A has integral entries also. If X is the column vector on the symbols $x_1, \cdots, x_n$ then the elements of the group corresponding to the rows of the column vector AX are a set of generators for the group. For it is clear that $X=A^{-1}AX$; hence the generators $x_1, \cdots, x_n$ may be expressed as integral combinations of the new generators.

BASIS THEOREM. *If G is an Abelian group generated by a finite number of elements, then G is the direct product of cyclic groups.*

Proof. Suppose every set of generators of G contains n or more elements. Then choose a set of n generators so that one of them, x_n, has minimal order k in the sense that no other set of n generators has an element of smaller order.

The other $n-1$ generators, $x_1, \cdots, x_{n-1}$, generate a proper subgroup H of G. By an induction argument on n we may assert that H is a direct product of $n-1$ cyclic subgroups. The theorem will be proved when it is shown that the intersection of H with the cyclic group C generated by x_n is zero. If this were not so then for some integers $a_n < k$, and $a_1, \cdots, a_{n-1}$, $a_n x_n = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$ or $a_1 x_1 + \cdots + a_{n-1} x_{n-1} - a_n x_n = 0$. If the greatest common divisor of $a_1, \cdots, a_n$ is d, then by the above theorem the element $g = (a_1/d)x_1 + \cdots + (a_{n-1}/d)x_{n-1} - (a_n/d)x_n$ is a member of a set of n generators of G of order a divisor of $d < a_n < k$. This is contrary to the choice of x_n. Accordingly we conclude that $H \cap C = 0$ as desired.

ON DIFFERENTIATION OF SERIES TERM-BY-TERM

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It is of course common knowledge that the convergence of a sequence of continuously differentiable functions on the unit interval does not imply the differentiability of the limit function nor the convergence of the derivatives of the functions in the sequence. We happened to notice that if the second derivatives of the functions in the sequence exist, and are uniformly bounded, then term-by-term differentiation is indeed valid. Since this simple fact does not seem to be widely known, nor in the literature, we offer this note.

The following lemma, adequate for our purposes, is a weakened version of results of Landau [3, 4] whose results in turn were stimulated by Littlewood's rediscovery [2, 5] of an observation of Moore ([6], p. 316.)

LEMMA 1. *Let g be a twice-differentiable real-valued function defined on an interval I of length l. Suppose that $|g(x)| \leq \epsilon$ and $|g''(x)| \leq k$ for all x in I, where ϵ and k are fixed positive numbers. Lastly, suppose $4(\epsilon/k)^{1/2} \leq l$. Then $|g'(x)| \leq 2(\epsilon k)^{1/2}$.*

Proof. Taylor's theorem gives $g(x+t) = g(x) + tg'(x) + t^2 \int_0^1 (1-y)g''(ty+x)dy$ for all x and $x+t$ in I. Therefore

$$|t|\,|g'(x)| \leq |g(x+t) - g(x)| + t^2 \int_0^1 (1-y)\,|g''(ty+x)|\,dy$$

$$\leq 2\epsilon + t^2 \int_0^1 k(1-y)dy = 2\epsilon + kt^2/2.$$

Consequently, $|g'(x)| \leq 2\epsilon/|t| + k|t|/2$ for all t such that x and $x+t$ are in I. For x in the first half of I, let $t = 2(\epsilon/k)^{1/2}$; for x in the second half of I, let $t = -2(\epsilon/k)^{1/2}$. Since $2(\epsilon/k)^{1/2} \leq l/2$, $x+t$ is in I. Using this t we get $|g'(x)| \leq 2(\epsilon k)^{1/2}$.

THEOREM 1. *Let f_n be a sequence of real-valued functions defined on a closed interval and satisfying* (i) *f_n converges uniformly; and* (ii) *f_n is twice differentiable, and for some constant k, $|f_n''(x)| \leqq k$ uniformly in n and x. Then f_n' converges uniformly.*

Proof. Let ϵ be a sufficiently small positive number. Then for all i and j sufficiently large, $|f_i - f_j| \leqq 2\epsilon$. Also, of course, $|f_i'' - f_j''| \leqq 2k$. We apply Lemma 1 to conclude that $|f_i' - f_j'| \leqq 4(\epsilon k)^{1/2}$.

COROLLARY. *The limit f of the sequence f_n is differentiable and f_n' converges to f' uniformly. Furthermore f' satisfies the Lipschitz condition $|f'(x) - f'(y)| \leqq k|x-y|$.*

The hypotheses of the theorem can be relaxed in three ways: first, one may replace (i) by the assumption that f_n converges at each point of a dense subset of the interval; second, one may replace (ii) by the hypothesis that f_n' exists everywhere and satisfies the Lipschitz condition $|f_n'(x) - f_n'(y)| \leqq k|x-y|$ uniformly in n, x and y; third and last, one need only assume that for some positive integer r, the $(r+1)$st derivatives exist and satisfy $|f_n^{(r+1)}| \leqq k$, or more generally that the rth derivatives exist and satisfy a uniform Lipschitz condition. That is:

THEOREM 2. *Let f_n be a sequence of real-valued functions defined on a closed interval and satisfying* (i) *f_n converges at each point of a dense subset of the interval; and* (ii) *for some $r \geqq 1$, the rth derivatives of the f_n exist and satisfy $|f_n^{(r)}(x) - f_n^{(r)}(y)| \leqq k|x-y|$ for all n, x, and y. Then for each j, $0 \leqq j \leqq r$, $f_n^{(j)}$ converges uniformly. (Here $f_n^{(0)}$ means f_n.)*

COROLLARY. *The limit f of the sequence f_n has r continuous derivatives, and for each j, $1 \leqq j \leqq r$, $f_n^{(j)}$ converges uniformly to $f^{(j)}$. Furthermore $f^{(r)}$ satisfies the Lipschitz condition $|f^{(r)}(x) - f^{(r)}(y)| \leqq k|x-y|$.*

Since the ideas needed for the proof of Theorem 2 are well known to analysts, we do not give a proof, but are content to suggest that a proof can be based on the circle of ideas involved in Ascoli's theorem which may be found, for example, in [1].

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THE MATRIX CONGRUENCE $X^2 \equiv I \pmod{p^a}$

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For p an odd prime, we determine all matrix solutions of order m of $X^2 \equiv I \pmod{p^a}$.* It is known ([1] p. 518) that the solutions of $X^2 \equiv I \pmod{p}$ are exactly the matrices similar modulo p to $J_t = \operatorname{diag}(I_t, -I_{m-t})$ with $0 \leqq t \leqq m$.

1. Any solution of

$$X^2 \equiv I \pmod{p^{a+1}} \tag{1}$$

is congruent modulo p^a to some root R of

$$X^2 \equiv I \pmod{p^a}, \tag{2}$$

where R in turn is congruent modulo p to some root Q of

$$X^2 \equiv I \pmod{p}. \tag{3}$$

To find all solutions of (1) which are congruent to R modulo p^a, we let $X = R + p^a T$ and seek all T modulo p such that $(R + p^a T)^2 \equiv I \pmod{p^{a+1}}$, or, equivalently, all T such that

$$\frac{R^2 - I}{p^a} + TQ + QT \equiv (0) \pmod{p}. \tag{4}$$

Now we choose $Q = J_t$ and we suppose R known. In solving (4) for

$$T \equiv \begin{bmatrix} A^{(t)} & B \\ C & D^{(m-t)} \end{bmatrix} \pmod{p},$$

we get equivalently

$$\begin{bmatrix} 2A & (0) \\ (0) & -2D \end{bmatrix} \equiv \frac{I - R^2}{p^a} \pmod{p}. \tag{5}$$

Now (5) is solvable for A and D modulo p and uniquely so if and only if null matrices of appropriate sizes occur in the upper right corner and the lower left corner of $p^{-a}(I - R^2)$ modulo p. This can be accomplished by replacing R by $R + p^a V$ with suitable V, if need be. Then, since (5) is independent of B and C, we see that there are exactly p^N solutions for T, where $N = 2t(m-t)$ is the number of entries in B and C. Since, moreover, for $Q = J_t$ and R fixed, incongruent solutions T_1, T_2 modulo p of (4) lead to incongruent solutions X_1, X_2 modulo p^{a+1} of (1), and incongruent solutions R_1, R_2 modulo p^a of (2) lead to incongruent solutions X_1, X_2 modulo p^{a+1} of (1), an induction argument shows us that there are exactly $p^{N \cdot a}$ solutions X of (1) for which $X \equiv J_t \pmod{p}$. In addition, (5) enables us to write down, albeit recursively, exactly what these solutions are.

* The referee has kindly pointed out that the number of solutions given in the present note is a special case of a result announced in [2].

2. Here we use the binomial theorem to obtain explicit, nonrecursive forms of the solutions discussed in Section 1.

Let

$$X \equiv \begin{bmatrix} E^{(t)} & F \\ G & H^{(m-t)} \end{bmatrix} \pmod{p^{a+1}},$$

where F and G are chosen arbitrarily modulo p^{a+1}, except that we take $F \equiv (0) \pmod{p}$ and $G \equiv (0) \pmod{p}$; and where we choose

$$(6) \quad E \equiv I - \tfrac{1}{2}FG - \tfrac{1}{8}FGFG - \tfrac{1}{16}FGFGFG - \tfrac{5}{128}FGFGFGFG - \cdots \pmod{p^{a+1}}$$

and

$$(7) \quad H \equiv -I + \tfrac{1}{2}GF + \tfrac{1}{8}GFGF + \tfrac{1}{16}GFGFGF + \tfrac{5}{128}GFGFGFGF + \cdots \pmod{p^{a+1}},$$

the coefficients in (6) and (7) being those in the expansion of

$$(1 - x)^{1/2} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \cdots.$$

We note first that there are only a finite number of terms in (6) and (7) since $F \equiv (0) \pmod{p}$ and $G \equiv (0) \pmod{p}$. We also observe from the arbitrary character of F and G that we have $p^{N \cdot a}$ choices for X modulo p^{a+1}. Clearly, each such X is congruent to J_t modulo p. Finally,

$$X^2 \equiv \begin{bmatrix} E^2 + FG & EF + FH \\ GE + HG & GF + H^2 \end{bmatrix} \pmod{p^{a+1}},$$

and this reduces by our choices above to $X^2 \equiv I \pmod{p^{a+1}}$.

3. The solutions of (1) which are yet to be considered are those built up from the matrices similar modulo p to some J_t, $0 \leqq t \leqq m$. We view them as transforms of the solutions we already have as follows: Let us fix t and take P an integral matrix with determinant prime to p. If now X is any solution of (1) which is congruent to J_t modulo p, then $P^{-1}XP$ modulo p^{a+1} is a solution of (1) which is congruent to $P^{-1}J_tP$ modulo p. Moreover, $X_1 \not\equiv X_2 \pmod{p^{a+1}}$ is equivalent to $P^{-1}X_1P \not\equiv P^{-1}X_2P \pmod{p^{a+1}}$. Thus there are exactly as many solutions of (1) which are congruent to $P^{-1}J_tP$ modulo p as there are solutions of (1) which are congruent to J_t modulo p, and we have just determined what they are.

4. The number of solutions of (1) is thus given by

$$\sum_{t=0}^{m} S_0(m, t)\, p^{2t(m-t)a},$$

where $S_0(m, t)$ is the number of matrices of order m which are similar modulo p to J_t. Finally, Hodges' paper ([1] pp. 518–519) gives $S_0(m, t) = g_m/g_t g_{m-t}$ where $g_t = \prod_{i=0}^{t-1} (p^t - p^i)$.

References

1. John H. Hodges, The matrix equation $X^2-I=0$ over a finite field, this MONTHLY, vol. 65, 1958, pp. 518–520.
2. J. E. Maxfield and M. Maxfield, The number of matrices belonging to a given integer (modulo n), abstract 558–10, Notices Amer. Math. Soc., June 1959, p. 288.

ON THE EVALUATION OF FINITE SUMS BY RESIDUES

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The evaluation of finite sums is of considerable importance in many branches of pure and applied mathematics. This note presents two theorems describing transformations of certain sums into residues expressions. These finite summation transformations are quite analogous to the Watson transformations [1] used for the evaluation of double-ended infinite series, and are related to the contour-integral summation schemes of Milne-Thomson [2].

THEOREM 1. *Suppose $a(z)$ has only a finite number of poles in the complex z-plane, all of which are simple and none of which are located at integers or at infinity. If these are the only singularities, then*

$$\sum_{n=0}^{N} a(n) = -\sum_{k=0}^{K} \zeta_N^{(1)}(z_k) \text{ Res. } \{a(z)\}_{z=z_k} + (N+1)\,a(\infty),$$

where the residues are evaluated at the poles of $a(z)$, and $\zeta_N^{(1)}(z) = \sum_{n=0}^{N} (z-n)^{-1}$.

The proof follows immediately from the contour integral

$$\oint_C a(z)\zeta_N^{(1)}(z)dz$$

taken around an infinitely large circle centered at the origin of the complex z-plane, since $\zeta_N^{(1)}$ is constructed with poles at the integers $0, 1, \cdots, N$ and residues equal to unity. For alternating series, a similar function with alternating residues leads to the following theorem.

THEOREM 2. *If $a(z)$ once again has only simple poles, not at integers or at ∞, then*

$$\sum_{n=0}^{N} (-1)^n a(n) = -\sum_{k=0}^{K} \zeta_N^{(2)}(z_k) \text{ Res. } \{a(z)\}_{z=z_k} + \tfrac{1}{2}[1+(-1)^N]a(\infty),$$

where the residues are evaluated at $a(z_k)=\infty$, and $\zeta_N^{(2)}(z) = \sum_{n=0}^{N} (-1)^n (z-n)^{-1}$.

The practical utility of these theorems depends on the number of poles of $a(z)$ being much less than the number of terms of the summation (*i.e.*, $K \ll N$), and is also dependent on the ease of finding these poles. These transformations then reduce the summation task to that of looking up a small number of values of the functions $\zeta_N^{(1)}$ and $\zeta_N^{(2)}$, which can be easily tabulated for the desired ranges

of z and N. We have denoted these functions by the Greek zeta because of their resemblance to incomplete Riemann zeta-functions of the first order, recalling that the complete Riemann function is defined by $\zeta(s, z) = \sum_{n=0}^{\infty} (z-n)^{-s}$. Some sketches of such incomplete zeta-functions are presented in Figure 1, and reveal the similarity to the common functions $\pi \cot \pi z$ and $\pi \csc \pi z$, respectively, due to the locations of the poles.

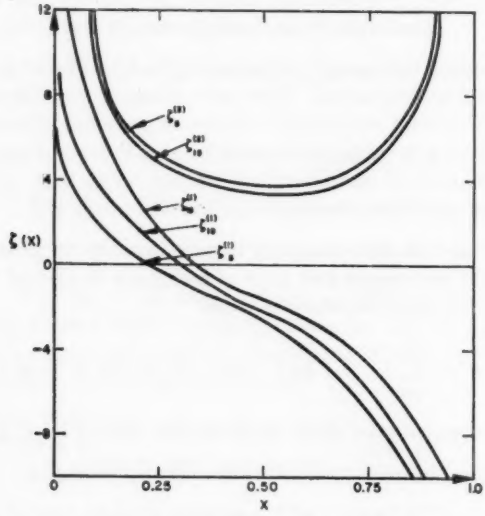

FIG. 1. Incomplete zeta-functions of first and second kinds.

As an application of the residues transformation, consider the following example:

$$\sum_{n=0}^{N} \frac{(n-a_1)(n-a_2)}{(n-b_1)(n-b_2)(n-b_3)} = -\frac{(b_1-a_1)(b_1-a_2)}{(b_1-b_2)(b_1-b_3)}\zeta_N^{(1)}(b_1) - \frac{(b_2-a_1)(b_2-a_2)}{(b_2-b_1)(b_2-b_3)}\zeta_N^{(1)}(b_2) - \frac{(b_3-a_1)(b_3-a_2)}{(b_3-b_1)(b_3-b_2)}\zeta_N^{(1)}(b_3),$$

where the a's and b's are distinct and nonintegral. The variation of this sum with increasing N is a function of the ζ-curves alone, and could be read directly from a zeta-table.

In conclusion, I should like to acknowledge gratefully the advice and encouragement of Dr. Aharon Ksienski.

References

1. P. Morse and H. Feshbach, Methods of Theoretical Physics, New York, 1953, p. 413.
2. L. M. Milne-Thomson, The Calculus of Finite Differences, New York, 1951, p. 223.

A GENERALIZATION OF THE FACTOR THEOREM

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A simple property of finite differences is used below to develop a necessary and sufficient condition for a polynomial $p(x)$ to be divisible by $(x-r)^m$. Since $p(x)$ is divisible by $(x-r)^m$ if and only if $p(rx)$ is divisible by $(x-1)^m$, only the case $r=1$ will be considered.

Let $F[x]$ consist of all polynomials in x with coefficients in a field F. If F has characteristic p, let m be a positive integer less than $p+1$.

THEOREM. *A polynomial* $p(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ *in* $F[x]$ *is divisible by* $(x-1)^m$ *if and only if*

$$a_0(n+1)^s+a_1n^s+a_2(n-1)^s+\cdots+a_{n-1}2^s+a_n=0 \quad \text{for } s=0,1,\cdots,m-1. \tag{1}$$

Necessity. Let $p(x)=q(x)(x-1)^m$ with $q(x)=\sum_{i=0}^{n-m}q_ix^{n-m-i}$ in $F[x]$. Since $(x-1)^m=\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{m-j}$, one has

$$p(x)=\left(\sum_{i=0}^{n-m}q_ix^{n-m-i}\right)\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{m-j}\right)$$

$$=\sum_{i=0}^{n-m}q_i\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{n-i-j}.$$

The condition (1) for $p(x)$ now becomes

$$\sum_{i=0}^{n-m}q_i\left[\sum_{j=0}^{m}(-1)^j\binom{m}{j}(n-i-j+1)^s\right]=0 \quad \text{for } s=0,1,\cdots,m-1. \tag{2}$$

For each i, the expression in brackets in (2) vanishes since it is an mth difference of x^s with $m>s$. Hence (2) holds and the necessity is established.

Sufficiency. The hypothesis now is that $p(x)$ satisfies (1) for $s=0,1,\cdots,m-1$. We use induction on m. The case $m=1$ follows from the Factor Theorem. Assume the theorem true for $m=k$. We wish to show that $p(x)$ is divisible by $(x-1)^{k+1}$ assuming that $p(x)$ satisfies (1) for $s=0,1,\cdots,k$ and (from the hypothesis of the induction) that $p(x)$ is divisible by $(x-1)^k$.

Divide $p(x)$ by $(x-1)^{k+1}$ obtaining

$$p(x)=q(x)(x-1)^{k+1}+r(x), \tag{3}$$

with $r(x)$ of degree not greater than k. Since $p(x)$ is divisible by $(x-1)^k$, $r(x)=a(x-1)^k$ with a in F.

The necessity part of the theorem shows that $q(x)(x-1)^{k+1}$ satisfies a condition such as (1) for $s=0,\cdots,k$. Since the condition (1) is linear and homogeneous in the coefficients a_i, the difference $r(x)$ of two polynomials which satisfy such a condition also does. The condition analogous to (1) with $s=k$ for

$r(x)=a\sum_{j=0}^{k}(-1)^j\binom{k}{j}x^{k-j}$ is

$$a\left[(k+1)^k-k\cdot k^k+\binom{k}{2}(k-1)^k-\cdots+(-1)^k\cdot 1^k\right]=0. \tag{4}$$

The expression within the brackets in (4) is $k!$ since it is a kth difference of x^k. If F has characteristic p, $k=m-1<p$. Hence $k!\neq 0$. Therefore $a=0$. Hence $r(x)=0$ and (3) shows that $p(x)$ is divisible by $(x-1)^{k+1}$. This completes the proof.

Finally, one may note that the theorem still holds if one replaces the powers $(n+1)^s, n^s, \cdots, 2^s, 1^s$ in (1) by the sth powers of any $n+1$ elements of F which are in arithmetic progression.

NORMS OF POLYNOMIALS

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Many investigations in number theory and Fourier series require integral estimations which are just somewhat beyond what the Schwarz inequality supplies. In this note we will break through the Schwarz barrier! This breakthrough, however, is so puny that it should be quite apparent that our main purpose is really to entice further research in this interesting and difficult subject.

Let $P(z)=z^n\pm z^{n-1}\pm\cdots\pm 1$, $n>0$ and let $\int|P|$ denote $(2\pi)^{-1}\int_0^{2\pi}|P(e^{i\theta})|\,d\theta$. The Schwartz inequality gives $\int|P|<\sqrt{(n+1)}$. We shall prove the

THEOREM. $\int|P|<\sqrt{(n+.97)}$.

Proof. Case 1: $\operatorname{Max}_{|z|=1}|P(z)|\leqq 2\sqrt{(\pi n)}$. Writing $P(e^{i\theta})=\sum_{k=0}^{n}\epsilon_k e^{ik\theta}$, we have

$$|P(e^{i\theta})|^2=P(e^{i\theta})P(e^{-i\theta})=\sum_{j=-n}^{n}\delta_j e^{ij\theta},$$

where $\delta_j=\epsilon_0\epsilon_{|j|}+\cdots+\epsilon_{n-|j|}\epsilon_n$. Now note that $\delta_0=n+1$ and that δ_j, being an integer and odd if $n-|j|$ is even, satisfies $|\delta_j|\geqq 1$ for $n-|j|$ even. Hence

$$\int|P|^4=\sum\delta_j^2\geqq(n+1)^2+2[(n+1)/2]\geqq(n+1)^2+n$$

and so, since $\int|P|^2=n+1$, we obtain $\int(|P|^2-(n+1))^2\geqq n$.

Now $|P|+\sqrt{(n+1)}\leqq 2\sqrt{(\pi n)}+\sqrt{(n+1)}\leqq(2\sqrt{\pi}+\sqrt{2})\sqrt{n}<5\sqrt{n}$ so that

$$\int(|P|-\sqrt{(n+1)})^2\geqq n/(5\sqrt{n})^2=\tfrac{1}{25};$$

again, since $\int|P|^2=n+1$, this shows that $\int|P|\leqq\sqrt{(n+1)}-1/\{50\sqrt{(n+1)}\}$. Hence $(\int|P|)^2\leqq n+1-\frac{1}{25}+\frac{1}{5000}<n+.97$.

Case II: $\operatorname{Max}_{|z|=1}|P(z)| = |P(e^{i\alpha})| = A\sqrt{n}$, $A > 2\sqrt{\pi}$. By Bernstein's theorem $|P'(z)| \leqq An^{3/2}$ for $|z| \leqq 1$. Hence, for θ in the interval

$$I = (\alpha - (2n)^{-1}, \alpha + (2n)^{-1}),$$

we have $|P(e^{i\theta})| \geqq A\sqrt{n}(1 - n|\theta - \alpha|)$. It follows that

$$B = \int_I |P(e^{i\theta})|^2 d\theta \geqq \tfrac{1}{2}A^2 > 2\pi.$$

Since $\int_{\text{comp } I} |P(e^{i\theta})|^2 d\theta = 2\pi(n+1) - B$, the Schwarz inequality gives

$$\int |P| \leqq (2\pi)^{-1}\sqrt{(B/n)} + \sqrt{(n + 1 - (2\pi)^{-1}B)}.$$

Now the right side is decreasing for $B \geqq 2\pi$ so that replacing B by 2π gives $\int|P| \leqq 1/\sqrt{(2\pi n)} + \sqrt{n}$ or

$$\left(\int |P|\right)^2 \leqq n + \sqrt{(2/\pi)} + 1/(2\pi n) \leqq n + \sqrt{(2/\pi)} + 1/(2\pi) < n + .97,$$

and the proof is complete.

It is clear from the proof that the constant .97 can be brought down slightly by modifying the numbers somewhat. To prove, however, that $\int|P| < \sqrt{n}$ for n large would call for a completely new approach. It has even been conjectured that $\int|P| \leqq c\sqrt{(n+1)}$, $c < 1$, but we are far from this result.

NOTE ON "A CLASS OF PARTIALLY ORDERED SETS"

Dr. Joseph Kruskal has informed the author of the Mathematical Note with the above title (this MONTHLY, vol. 67, 1960, pp. 448–449) that the results are not new. They have previously appeared, with different terminology, in G. Higman, *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc., vol. 2, 1952, pp. 326–336; in R. Rado, *Partial well-ordering of sets of vectors*, Mathematika 1, 1954, pp. 89–95; and in J. Kruskal's thesis (unpublished), Princeton, 1952. Moreover, they are making yet another appearance in J. Kruskal, *Well quasi-ordering, the tree theorem, and Vazsonyi's conjecture*, Trans. Amer. Math. Soc., vol. 95, 1960, pp. 210–225.

CLASSROOM NOTES

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A NOTE ON δ AND ϵ

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Our freshmen, at least some of them, can solve the problem: $\lim_{x\to 2} 3x=6$, by showing that if $\delta=\frac{1}{3}\epsilon$, then $0<|x-2|<\delta$ implies $|3x-6|<\epsilon$, for any positive number ϵ. But confront them with $\lim_{x\to 2} x^2=4$ and they haven't the remotest idea how to proceed. Of course one can write $|x^2-4|=|x-2|\cdot|x+2|$ and start by restricting x to values between 1 and 3 so that $|x+2|<5$, but this leaves me cold. The purpose of this note is to exhibit a systematic technique for attacking this and similar problems, applicable in fact to all polynomials.

Considering the problem at hand, *i.e.*, $\lim_{x\to 2} x^2=4$, we wish to exhibit a positive number δ such that if $0<|x-2|<\delta$, then $|x^2-4|<\epsilon$ for every positive ϵ. The whole trick is to expand x^2-4 in powers of $x-2$. Thus $x^2-4=(x-2)^2+4(x-2)$. Then

$$|x^2-4| = |(x-2)^2+4(x-2)| \leqq |x-2|^2+4|x-2|.$$

The latter will be less than ϵ provided that (completing the square)

$$|x-2|^2+4|x-2|+4<\epsilon+4$$

or

$$|x-2|+2<\sqrt{(\epsilon+4)} \text{ or } |x-2|<\sqrt{(\epsilon+4)}-2.$$

Thus, choosing $\delta=\sqrt{(\epsilon+4)}-2$, the problem is solved.

Similarly, to prove that $\lim_{x\to 2} x^3=8$, one chooses $\delta=\sqrt[3]{(\epsilon+2^3)}-2$. In general, for $a>0$, to prove $\lim_{x\to a} x^n=a^n$, one chooses $\delta=\sqrt[n]{(\epsilon+a^n)}-a$. Obviously, such a δ is always positive.

If $a<0$ for the case $\lim_{x\to a} x^n=a^n$, we choose $\delta=\sqrt[n]{(\epsilon+|a|^n)}-|a|$; this choice of δ will, of course, work for $a>0$ as well.

For a problem such as $\lim_{x\to 2} 5x^2=20$, since $|5x^2-20|<\epsilon$ if $|x^2-4|<\frac{1}{5}\epsilon$, we choose $\delta=\sqrt{(\frac{1}{5}\epsilon+4)}-2$, and so on. In the case of a general polynomial this technique is applied to the individual terms.

As a simple illustration consider $\lim_{x\to 1}(x^3+5x^2)=6$. We wish to exhibit a positive number δ such that if $0<|x-1|<\delta$, then $|x^3+5x^2-6|<\epsilon$. Now

$$|x^3+5x^2-6| = |x^3-1+5x^2-5| \leqq |x^3-1|+5|x^2-1|.$$

Also, $|x^3-1|<\frac{1}{2}\epsilon$ if $0<|x-1|<\delta_1$, where $\delta_1=\sqrt[3]{(\frac{1}{2}\epsilon+1)}-1$; and $5|x^2-1|<\frac{1}{2}\epsilon$ if $0<|x-1|<\delta_2$, where $\delta_2=\sqrt{(\frac{1}{10}\epsilon+1)}-1$. Hence, if we choose δ to be the smaller of δ_1 and δ_2, the problem is solved.

FINDING THE CHANGE OF VARIABLE CARRYING ONE FINITE INTERVAL ONTO ANOTHER

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It is often required to find a linear change of variable, $t=px+q$ (*), in such a way that a given interval $a<x<b$ is mapped onto a prescribed interval $\alpha<t<\beta$. One way is to substitute into(*) and solve for p and q. Another way, motivated by the fact that the ratio of the lengths of the intervals is $(\beta-\alpha)/(b-a)$, is displayed in the following inequalities:

$$a < x < b, \qquad 0 < x - a < b - a,$$

$$0 < \frac{\beta - \alpha}{b - a}(x - a) < \beta - \alpha,$$

$$\alpha < \frac{\beta - \alpha}{b - a}(x - a) + \alpha < \beta.$$

If $\alpha=-\beta<0$ (for example, we often wish $-\pi<t<\pi$ in Fourier analysis), a still simpler process exploits the fact that the midpoint of the given interval is to be mapped onto the origin:

$$a < x < b,$$

$$-\frac{b - a}{2} < x - \frac{a + b}{2} < \frac{b - a}{2},$$

$$-\beta < \frac{2\beta}{b - a}\left(x - \frac{a + b}{2}\right) < \beta.$$

Suppose, for illustration, that we are given the interval $-1<x<4$. The inequalities

$$-\frac{5}{2} < x - \frac{3}{2} < \frac{5}{2}, \qquad -\pi < \frac{2\pi}{5}\left(x - \frac{3}{2}\right) < \pi$$

lead us to put $t=2\pi(x-3/2)/5$. Alternatively, the inequalities

$$0 < x + 1 < 5, \qquad 0 < \frac{2\pi}{5}(x + 1) < 2\pi$$

suggest a variable $u=2\pi(x+1)/5$ running over the interval from 0 to 2π.

Experiments indicate that these extremely easy manipulations can be performed quite rapidly in numerical examples. The explicit translation of the origin and change of scale seem enlightening to students.

THE TRIANGULAR INEQUALITY IN THE PROJECTIVE MODEL OF A HYPERBOLIC GEOMETRY

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The triangular inequality is the main obstacle if one wishes to present Klein's model of a hyperbolic geometry in a synthetic way. Once this inequality is established, the rest of the way is smooth.

As I could not find this in any text book, I give here the proof of a theorem (Theorem 3) from which the triangular inequality follows in a well-known way. As was to be expected, the relation to be proved has to be found equivalent to a relation of separation. This separation is the one stated in Theorem 2 and makes use of the fact that a point is assumed to be an interior point.

I adopt the notations used in H. S. M. Coxeter, *The Real Projective Plane*, Cambridge, 1955, and assume the knowledge of the following definitions and theorems:

1. *Given two projectively related ranges a and b, $A \in a$, $B \in b$, $A \overline{\wedge} B$† then the lines AB envelop a conic touching a and b.*

2. *Given a conic by two projectively related ranges a and b and a point P on the line joining the points of contact of a and b with the conic. Then the conic associates an involution to P in such a way that if $A_\mu A_\nu \overline{\wedge} B_\mu B_\nu$, in the projectivity $a \overline{\wedge} b$ then PA_μ, PB_μ, PA_ν, $PB_\nu \overline{\wedge} PB_\mu$, PA_μ, PB_ν, PA_ν.*

3. *The point P is interior point of the conic if and only if this involution is elliptic.*

4. *An involution $ABCD \overline{\wedge} BADC$ is elliptic if and only if $AB \| CD$.*

5. *Two chords of a conic are perpendicular if they are on lines conjugate with respect to the conic.*

6. *If A, B are two points inside a conic c and if PQ is the chord containing them we put $\|AB\| = |\log \mathfrak{R}(PQ, AB)|$, where $\mathfrak{R}$ denotes the cross ratio.*

7. *A harmonic homology with center A and axis a is a 1-1 mapping of the projective plane such that if $P \to P'$ by this homology then A, P, P' are collinear and $H(AA_0, PP')$ where $A = PP_0' \cdot a$. By a harmonic homology, lines are mapped into lines.*

8. *If center and axis of a harmonic homology are pole and polar with respect to a conic c then this homology has the conic as an invariant in such a way that interior points are mapped into interior points. If A, B are interior points, $A \to A'$, $B \to B'$ by this homology then $\|AB\| = \|A'B'\|$.*

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† The symbol $\overline{\wedge}$ denotes a projectivity.

THEOREM 1. *Let $ABCD$ be four points on a conic c. Put $AB\cdot CD=Q$, $AD\cdot BC=R$, $QR=p$. Denote by a, b, c, d the tangents on c in A, B, C, D, put $bd=E$, $ac=E^*$. Then $EE^*=QR$ and $H(QR, EE^*)$.*

Proof. p is the polar of $P=AC\cdot BD$. This proves $EE^*=QR$. The harmonic homology with Q as center, PR as axis has c as an invariant and $E\to E^*$ by this homology. This proves the theorem.

THEOREM 2. *Let A, B, C, D be four points on a conic c, let $P=AC\cdot BD$, $Q=AB\cdot DC$, $R=AD\cdot BC$ be the vertices of the diagonal triangle of the quadrilateral $ABCD$. Put $p=QR$; $S=AC\cdot p$, $T=BD\cdot p$. If b is tangent on c in B and $E=bp$ then P is interior point of c if and only if $QR\|ET$. If $E=Q$ viz. $E=R$ then the conic degenerates.*

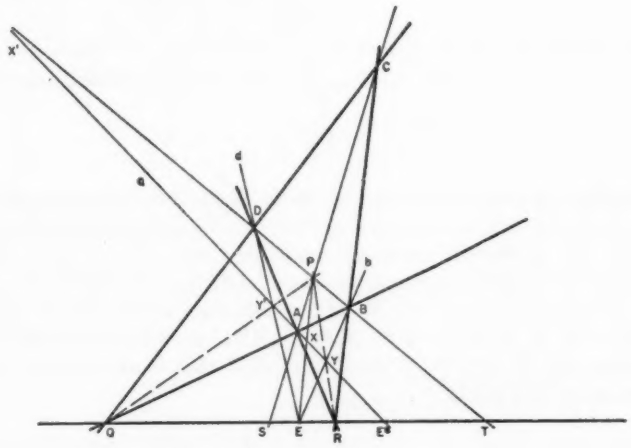

FIG. 1

Proof. (Fig. 1) Let a, d be the tangents on c in A viz D. It is $bd=E$. Furthermore, if $ap=E^*$ then $H(QR, EE^*)$ (Theorem 1). Put $F=PR\cdot AB$, $Y=ab$. As $H(QR, ST)$ and $QRST\overset{P}{\barwedge}QFAB$§ it is $H(QF, AB)$. Therefore $QREE^*\barwedge QFBA$. This being a perspectivity it follows F, Y, R *viz.* P, Y, R are collinear points. In the same way P, Y', R are collinear where $Y'=ad$. Put $X=PE\cdot a$, $X'=BD\cdot a$. Then $X'Y'XY\overset{P}{\barwedge}TQER$.

Consider c being generated by the two projective ranges b and d. The pairs of points, namely YY' and ED are corresponding points. The involution associated to P by c is therefore determined by the pairs PY, PY' and $PE=PX$, $PD=PX'$. It is elliptic if and only if $XX'\|YY'$ and as $XX'YY'\barwedge ETRQ$ it is elliptic if and only if $ET\|RQ$. On the other hand a point is interior point of a conic if and only if the involution associated to it is elliptic.

§ The symbol $\overset{P}{\barwedge}$ denotes a perspectivity from P.

If $E=R$ or $E=Q$ then the tangent b has two distinct points in common with c and thus the theorem is proved.

THEOREM 3. *If ABC is a triangle inside a conic c, rectangular at C then $\|AB\|>\|AC\|$.*

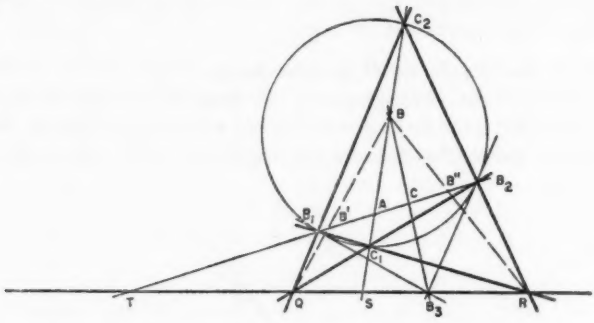

FIG. 2

Proof. (Fig. 2) Let AC be on the chord B_1B_2, AB on the chord C_1C_2. Put $B_1C_2\cdot B_2C_1=Q$, $B_1C_1\cdot B_2C_2=R$, $QR=p$, $B_1B_2\cdot p=T$, $C_1C_2\cdot p=S$. If $B'=B_1B_2\cdot BQ$, then $B\to B'$, $A\to A$ by the harmonic homology with Q as center, AR as axis, therefore $\|AB\|=\|AB'\|$. In the same way, $\|AB\|=\|AB''\|$ with $B''=B_1B_2\cdot BR$. Denote b_1, b_2 the tangents in B_1, B_2. If $B_3=b_1b_2$ then $B_3\in p$ and $B_3T\|QR$ according to Theorem 2. It is $C=BB_3\cdot B_1B_2$ as ABC is rectangular at C. If $\|AC\|>\|AB\|$ then $AC\|B'B''$. As $B'ACB''\overset{B}{\underset{\wedge}{=}}QSB_3R$ this is equivalent to $SB_3\|QR$. This contradicts $B_3T\|QR$.

EVALUATION OF DIRICHLET'S INTEGRALS

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The following method of evaluating the well-known multiple integral

$$I_n=\int\cdots\int_{x_k\geq 0,\,\Sigma x_k\leq 1}(1-x_1-\cdots-x_n)^{\alpha_0-1}x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}dx_1\cdots dx_n$$

for constant $\alpha_k>0$ in terms of the gamma function may be interesting. In the case $n=1$, it gives the evaluation of the beta function [cf. 1, 2, 3].

Consider the integral

$$g_{n+1}=\int\int\cdots\int_{\xi_k\geq 0,\,\Sigma\xi_k\leq\omega}\xi_0^{\alpha_0-1}\xi_1^{\alpha_1-1}\cdots\xi_n^{\alpha_n-1}f(\xi_0+\xi_1+\cdots+\xi_n)d\xi_0d\xi_1\cdots d\xi_n$$

with constant $\alpha_k > 0$ and $\omega > 0$, the function f being such that $\int_0^\omega x^{\alpha-1} f(x)dx$ exists for every $\alpha > 0$. Let us change the variables of integration by putting

$$\xi_0 = x_0(1 - x_1 - \cdots - x_n), \qquad \xi_1 = x_0 x_1, \cdots, \xi_n = x_0 x_n.$$

The Jacobian of this transformation is easily computed. Observing that $\sum \xi_k = x_0$, one gets, by just an addition of rows,

$$\frac{\partial(\xi_0, \xi_1, \cdots, \xi_n)}{\partial(x_0, x_1, \cdots, x_n)} = \frac{\partial(x_0, \xi_1, \cdots, \xi_n)}{\partial(x_0, x_1, \cdots, x_n)} = x_0^n.$$

Hence we find that

$$g_{n+1} = I_n \int_0^\omega x_0^{\alpha_0+\alpha_1+\cdots+\alpha_n-1} f(x_0)dx_0. \tag{1}$$

Here if we take $f(x) = e^{-x}$ and let $\omega \to \infty$, we at once obtain the desired expression of I_n in terms of the gamma function:

$$I_n = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)}. \tag{2}$$

At the same time, on inserting (2) into (1) and changing the notation slightly, we have another familiar formula:

$$\int \cdots \int_{x_k \geq 0, \Sigma x_k \leq \omega} x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} f(x_1 + \cdots + x_n)dx_1 \cdots dx_n$$

$$= \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n)} \int_0^\omega x^{\alpha_1+\cdots+\alpha_n-1} f(x)dx. \tag{3}$$

The above procedure has been motivated by the simple observation that a check of (3) can be obtained by choosing $f(x) = e^{-x}$ and letting $\omega \to \infty$.

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LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

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It is well known that the homogeneity of an expression depends upon the many ways that the dimensions or degrees are assigned to its variables; consequently, there are several ways in which an expression can be homogeneous. Subsequent to the assignment of degrees, an expression is said to be homogeneous if each term of its sum has the same degree.

Professor H. B. Phillips points out in his text* that a differential equation which is not homogeneous, say in x and y, can sometimes be made so by assigning dimension 1 to x and its differential dx and at the same time assigning dimension n to y and its differential dy. When an equation is made homogeneous by assigning dimensions in this manner the substitution $y=vx^n$ will render the new equation in v and x separable. It is also pointed out that the only important aspect of the substitution $y=vx^n$ is that the variable v is of dimension zero. Many types of equations usually solved by other methods can also be solved by this "homogeneous" method. In order to determine n in the substitution $y=vx^n$, let x and its differential dx be of dimension 1 in the original equation, and replace y by u^n and dy by u^n, where u is of dimension 1. Some examples follow.

Exact differential equation: $(x^2+3xy^2)dx+(3x^2y-2y^5)dy=0$. The above consideration yields

$$(x^2 + 3x^1u^{2n})dx^1 + (3x^2u^n - 2u^{6n})u^n = 0.$$

The terms are respectively of dimensions 3, $2+2n$, $2+2n$, $6n$. For each term to be of the same dimension, we must have $3=2+2n$, whence $n=1/2$. Therefore $y=vx^{1/2}$ will separate variables.

An integrating factor, $1/xy$: $(x^2y+x)dy+(xy^2-3y)dx=0$.

$$2 + 2n,\ n + 1,\ 2n + 2,\ n + 1.$$

Then $2n+2=n+1$. Here $n=-1$ and the substitution $y=v/x$ will separate variables.

Linear equation in y: $3dy/dx+5y/x=x^2$.

$$n + 1,\ n + 1,\ 4; \qquad n + 1 = 4, \quad \text{whence} \quad y = vx^3.$$

Bernoulli's equation: $xdy/dx+2y=3x^2y^{4/3}$.

$$n + 1,\ n + 1,\ 4n/3 + 4.$$

Then $4n/3+4=n+1$, $n=-9$; $y=vx^{-9}$.

Another type: $(x^{1/3}+x^{5/3}y^{1/5})dx+(x^{32/9}y^{-2/3}+x^{22/9}y^{-5/6})dy=0$.

$$4/3, \quad 8/3 + n/5, \quad 32/9 + n/3, \quad 22/9 + n/6;$$

$$8/3 + n/5 = 4/3, \quad n = -20/3, \quad \text{whence} \quad y = vx^{-20/3}.$$

It is observed that the dimension of each term must be obtained before determining whether a unique n can be found.

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* Differential Equations, New York, 1951.

THE CHAIN RULE

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The average calculus student finds difficulty with problems requiring use of the chain rule for differentiation of functions of several variables because he is unable to produce with confidence the correct formula which should be applied. The proof of the chain rule formula, using the continuity of the first partials to obtain an alternative form of Δf (the change in the function f), is too complicated and deep for most students at this level to use as an aid in writing the proper formula. The purpose of this note is to present a scheme which has the properties of a successful mnemonic: sufficiently "natural" and simple to be easily recalled, and quickly applied.

First, it is necessary to be confident of what derivatives make sense, and whether these be partial or ordinary derivatives. A function-variable diagram (or function diagram)* is quite adequate for this task. Assuming that we are concerned with functions $w=w(x_1, \cdots, x_r)$, $x_i=x_i(y_1, \cdots, y_s)$, $i=1, \cdots, r$ and $y_j=y_j(z_1, \cdots, z_t)$, $j=1, \cdots, s$, we construct the following diagram:

$$w$$

$$x_1, \cdots, x_r$$

$$y_1, \cdots, y_s$$

$$z_1, \cdots, z_t$$

The significance of the diagram is not only that w is a function of the x's, and that w is a function of the y's, and also of the z's, but that each variable is a function of the variables in any row lower than that in which the given variable occurs. Thus it makes sense to talk of the derivative of any variable with respect to any variable which appears lower on the function diagram.

A diagram

$$f$$

$$x \quad u \quad v$$

$$x \quad y$$

has special dangers. We can differentiate with respect to x, where f is thought of as a function of x, u and v, or where f is thought of as a function of x and y. It seems preferable, at first, not to use the language of the last sentence but rather to speak of differentiating with respect to the x in the second row or the x in the third row. Of the many possible ways of distinguishing between these derivatives; the usual one $(\partial f/\partial x)_{u,v}$ and $(\partial f/\partial x)_y$ is easily accepted and understood. Finally, the derivative with respect to any variable is ordinary or partial

* Such diagrams are presented in many calculus texts, *e.g.*, G. B. Thomas, Jr., Calculus and Analytic Geometry, Reading, Mass. 1953, p. 503. However it seems preferable to emphasize by the terminology the functions which are implied by such a diagram.

according to whether the variable with respect to which we differentiate is alone in the row, or occurs with other variables.

All of the preceding is rather standard, but the following method for obtaining the chain rule seems to be novel. In the first function diagram, suppose we wish to find $\partial w/\partial z_1$. We define a path starting at w and ending at z_1 to be a sequence w, x_i, y_j, z_1 where $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$, and we associate with such a path the product $(\partial w/\partial x_i)(\partial x_i/\partial y_j)(\partial y_j/\partial z_1)$, assuming r, s, t, are all >1. The chain rule states that $\partial w/\partial z_1$ is equal to the sum of the products associated with all possible paths which start at w and end at z_1.

Thus, to obtain the chain rule formula, we first construct a function diagram, and then enumerate all possible paths* which start at the dependent variable and end at the independent variable. The chain rule states that the derivative is equal to the sum of the products associated with all the paths just enumerated.†

The path concept of the chain rule not only assures us of writing the correct formula but also suggests, in some cases, when substitutions are required. If we are to show that $f(x-at)$ satisfies the one-dimensional wave equation $\partial^2 f/\partial t^2 = a^2(\partial^2 f/\partial x^2)$ we note that the paths of the diagram $\frac{f}{x\ t}$ starting at f and ending at x and t have nothing in common, and thus it is not surprising that the derivatives cannot be related without putting another row between f and x, t. (Indeed, it is clear that the only additional rows which can possibly be of aid are those *between* f and x, t.) Although there are many other profitable returns which this visual picture of the chain rule yields, merely its success as a mnemonic is sufficient to warrant its consideration.

In conclusion, we find that given the function diagram, everyone is able to write easily and quickly the chain rule formulas, and that most students are able to produce the appropriate diagram when all their thoughts are concentrated on this effort alone.

MATRIX SLIDE RULES

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We consider a k by ∞ matrix A with columns the vectors C_n, $-\infty < n < +\infty$ whose k elements $c_{1,n}, \cdots, c_{k,n}$ are defined as follows: The k adjacent vectors $C_0, C_1, \cdots, C_{k-1}$ are the successive columns of the identity matrix of order k, and all other $c_{j,n}$ are defined recursively by the relation

$$c_{j,n} = \sum_{m=1}^{k} a_m c_{j,n-m}, \qquad a_k \neq 0. \tag{1}$$

This means in particular that the vector C_k has elements $c_{j,k} = a_{k-j+1}, j = 1, \cdots, k$.

* For a general function diagram, we define a path as a sequence of variables, one and only one from each row in turn, starting with the dependent and ending with the independent variable.

† The product associated with a path is the product of the derivative of each variable in the sequence with respect to its successor.

Let A_j, $-\infty < j < \infty$, be the k by k submatrix of A consisting of the consecutive column vectors C_j, C_{j+1}, $\cdots$, C_{j+k-1}. We note readily that $A_0 = I_k$ and A_1 is a nonsingular matrix with characteristic equation

$$x^k = \sum_{m=1}^{k} a_m x^{k-m}. \tag{2}$$

It may be easily established by induction that $A_j = (A_1)^j$, $-\infty < j < \infty$. In the special case with all the a_j integers and $a_k = 1$ it is obvious that all elements of A are integers.

We now see how finite submatrices of A may be used to construct matrix slide rules which are helpful in suggesting numerous problems involving the matrices A_j and providing a quick check for the answers to these problems. We may, for instance, display in equally spaced columns on our slide rule the $2n+k$ adjacent columns C_{-n}, C_{-n+1}, $\cdots$, C_0, $\cdots$, C_{n+k-1} of A. By sliding across this rule a cover concealing all but a consecutive grouping of k columns we can display in turn the matrices $(A_1)^{-n}$, $(A_1)^{-n+1}$, $\cdots$, $(A_1)^0$, A_1, $\cdots$, $(A_1)^n$. The names of the various matrices thus displayed can be written in a row along the upper part of the rule so that an auxiliary opening will show which matrix is being displayed at a given time.

If one wants the inverse of a given matrix $A_j = (A_1)^j$, $-n \leqq j \leqq n$, one has only to locate the matrix A_{-j} on the slide rule. Should one wish to know the product of A_j and A_m with $-n \leqq j, m, j+m, \leqq n$ one needs only to locate A_{j+m}. Multiplication of any A_j and A_m is commutative, which suggests another type problem, that is, verifying this fact for particular values of j and m. We have excluded the case $a_k = 0$ in order that our base matrix A_1 might have an inverse, although with $a_k = 0$ it would still be possible to display the positive powers of A_1 in a semi-infinite matrix A corresponding to the columns C_0, C_1, $\cdots$, C_k, $\cdots$.

We see from (2) that A_1 may be constructed to have any desired eigenvalue set $\lambda_1, \cdots, \lambda_k$ where $0 < |\lambda_1| \leqq \cdots \leqq |\lambda_k|$. By extending C_j with a space followed by the array $(\lambda_1)^j, \cdots, (\lambda_k)^j$ and making an appropriately labeled auxiliary opening in our cover we can display the eigenvalues of the matrix with the matrix itself. When $|\lambda_k/\lambda_1| > 1$ we can pick j such that the P-condition number of A_j is as large as we please. Applying this technique when $k = 2$ and $a_1 = a_2 = 1$ leads to the 2 by 2 matrices of the author's paper, *Generalized Fibonacci numbers and associated matrices*, pages 745–752 in this issue of the MONTHLY. However for $k > 2$ the ill conditioned matrices of the other paper differ from those obtained here. We also note that verification that a particular matrix A_1 satisfies its characteristic equation (2) is immediately evident from the fact that all elements of A_k are defined recursively by (1).

We illustrate the above remarks by considering the matrices displayed below and commenting upon them. If $k = 4$, $a_j = 1$, $j = 1, 2, 3, 4$, and n is taken as 7, the 18 columns from C_{-7} through C_{10} suffice to display the matrix B (enclosed with

its label in the rectangular polygon representing the cover opening of the rule) and its integral powers B^j, $|j| \leqq 7$. We note that the elements $c_{1,j}$ in this case are the generalized Fibonacci numbers $f_{j,4}$ of the author's paper mentioned above, and that $c_{4,j}=f_{j+1,4}$. However the $c_{2,j}$ and $c_{3,j}$, although determined by the same recursion relation, are not numbers of the same set because the initial values are ordered differently.

B^{-7}	B^{-6}	B^{-5}	B^{-4}	B^{-3}	B^{-2}	B^{-1}	I_4	B	B^2	B^3	B^4	B^5	B^6	B^7
0	1	−3	2	0	0	−1	1	0	0	0	1	1	2	4	8	15	29
4	1	−2	−1	2	0	−1	0	1	0	0	1	2	3	6	12	23	44
−4	5	−2	0	−1	2	−1	0	0	1	0	1	2	4	7	14	27	52
1	−3	2	0	0	−1	1	0	0	0	1	1	2	4	8	15	29	56

By using a permutation matrix such as D below we could easily construct a cylindrical slide rule from which any integral power of D, say D^j, might be obtained by rotating j columns of the cylinder past the opening in the cover, when $D^0=I$ was there initially. Since the eigenvalues of D and its powers are readily expressed algebraically in terms of the cube roots of unity, we have indicated in this case how the eigenvalues for a particular power of D may be displayed along with the matrix itself. One needs only to mount the matrix below on a cylinder of circumference nine column widths and fit it with a cover opening as indicated.

	D^{-3}	D^{-2}	D^{-1}	D^0	D	D^2	D^3	
1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
	1	1	1	1	1	1	1	
	1	w	w^2	1	w	w^3	1	
	1	w^2	w	1	w^2	w	1	

As our last example we display matrices T^j, $0 \leqq j \leqq 4$, where T^j has P-condition number 10^j, and simultaneously display the eigenvalues for T^j.

	I_3	T	T^2	T^3	T^4	
1	0	0	−10	−100	−1010	−10100
0	1	0	1	0	1	0
0	0	1	10	101	1010	10101
	1	−1	1	−1	1	
	1	1	1	1	1	
	1	10	100	1000	10000	

We note that T^n has eigenvalues $(-1)^n$, 1, and 10^n so that its P-condition number (ratio of the largest to smallest absolute values for its eigenvalues),

10^8, is also displayed as the lowest framed number. To illustrate the instability of a system with coefficient matrix of high condition number we consider the following system with its coefficient matrix T_3 having P-condition number 1,000. We suppose all coefficients in this system are exactly known except that of y in the third equation whose error does not exceed 1%. Thus we are really concerned with possible solutions of $s(e)$ for $|e| \leqq 1.01$ where

$$s(e) \sim \begin{aligned} -10x - 100y - 1010z &= -10{,}100 \\ x \qquad\quad + \qquad z &= 0 \\ 10x + (101 + e)y + 1010z &= 10101 \end{aligned}$$

We note that the exact solutions of $s(e)$ within the 1% error range include the widely scattered vectors

$$\begin{aligned} e = 0: &\quad (x, y, z) = (-10, 1, 10), \\ e = -.99: &\quad (x, y, z) = (-.1, 100, .1), \\ e = -1.01: &\quad (x, y, z) = (-20.1, -100, 20.1), \\ e = -1: &\quad \text{no solution.} \end{aligned}$$

We conclude with a few suggestions that simplify the construction of matrices of the above type, should anyone wish to make one for his own amusement or edification. After choosing k and the set $\{a_k\}$ occurring in column C_k, one fills in columns $C_0, C_1, \cdots, C_k$ accordingly. It is now convenient to write on a separate thin strip of paper spaced according to the column width used the symbols

$$\underset{\cdot}{a_k} + \underset{\cdot}{a_{k-1}} + \cdots + \underset{\cdot}{a_1}. \tag{3}$$

Each element of C_{k+1} is now readily computable by placing the strip (3) above its k predecessors and carrying out the indicated operations. Other columns to the right of C_{k+1} can be filled in by successive applications of this technique. Since, when $a_k \neq 0$, the recursion relation (1) is solvable for a given $c_{j,n}$ in terms of $c_{j,n+p}$, $p = 1, \cdots, k$, entries for columns C_{-1} may be computed with the aid of a strip like (3) which takes the form

$$(-\underset{\cdot}{a_{k-1}}/\underset{\cdot}{a_k}) + (-\underset{\cdot}{a_{k-2}}/\underset{\cdot}{a_k}) + \cdots + (-\underset{\cdot}{a_1}/\underset{\cdot}{a_k}) + (\underset{\cdot}{1}); \tag{4}$$

entries for columns $C_{-2}, C_{-3}, \cdots$, etc., may then be obtained in the same manner. A final check of a matrix such as the first illustration above, would be to show that the left-most k by k submatrix, B_{-7}, is the inverse of the rightmost, B_7, by showing that their product is the identity.

Epilog. The general term of any sequence defined recursively by (1) may be expressed in terms of the row vector R: $(r_0, r_1, \cdots, r_{k-1})$ and the column vectors C_j as follows: $r_n = R \cdot C_n$.

Postscript. For an amplification of the theory underlying the results of this note, see Ralph W. Pfouts and C. E. Ferguson, *A matric general solution of linear difference equations with constant coefficients*, Math. Mag., vol. 33, 1960, pp. 119–127.

MATHEMATICAL EDUCATION NOTES

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THE NASDTEC-AAAS TEACHER PREPARATION AND CERTIFICATION STUDY

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The remarkable array of letters in the title stands for the National Association of State Directors of Teacher Education and Certification, and for the American Association for the Advancement of Science. The various state departments of education, through their directors of certification, are attempting to work out a new approach to certification, to meet the problems stated below. Since there is great urgency about mathematics and the sciences, they are beginning with these fields, with the help of the AAAS, and with a grant from the Carnegie Corporation of New York.

The problems. Given our present school and university system, training programs for the secondary teacher must satisfy several boundary conditions. First, the teacher should be equipped to begin teaching after four years of college. This condition is imposed by the beginning teacher, or perhaps rather by the beginning salaries in the schools. Thus there are only 120–130 semester hours available. Since some of this time will be taken up with requirements peculiar to the individual colleges—courses in religion, unusually large requirements in language or some other field—room must be left for these in any general requirements for certification.

Second, the teacher must be prepared to teach two or three subjects. The primary reason for this is the very large number of high schools that are too small for proper specialization. Conant* estimates that 17,000 out of the 21,000 high schools in the country are too small for this. By and large, there is nothing a state department of education can do about the too-small high school except try exhortation. Amalgamation is, generally, up to the voters in the individual school districts. Thus, in one state there are two towns five miles apart, each with its own rather small high school. One combined school half-way between could provide a vastly better program, but amalgamation would end the annual football game between the two schools, and nothing can make the voters give that up. Since small high schools will continue to exist for some time, most states are forced into requiring a minimal preparation in other fields than the major.

Third, the teacher must be trained in teaching methods, by courses in education. I must admit certain prejudices here, but I must also admit to very little

* James B. Conant, The Child, the Parent, and the State, Harvard University Press, 1959, Notes, p. 161 *et seq.*

experience. Certainly, some training is called for, and as of 1957, minima* ranged from 12 to 27 semester hours, with a median of 18 hours and an average of 19−. The requirements in this area are often given in great detail.

Fourth, an increasing number of states†—28 in 1959—have a "general education" requirement. To illustrate this, Kansas requires 10 hours of social science; 10 hours of literature, language, art, philosophy, or general religion; 10 hours of physical and biological sciences; and 15 hours of electives in these areas, making 45 semester hours. A student taking a normal bachelor's degree in a good arts college would almost certainly satisfy these requirements in satisfying his college's graduation requirements. The students who would not meet it are the ones from weak colleges with a minimal subject-matter preparation and with a vast number of courses in education, typing, and bookkeeping, or the ones with degrees in hotel management or mobile homes. To prevent these students from becoming teachers is the intent of the requirement, but it and arithmetic show that there is no easy solution.

Fifth, many schools have special requirements that have been imposed by the legislatures—Washington State history, or the Constitution of Texas, or the harmful effects of alcohol, for example.

With all these conditions, there is not much room to move around in. What has actually grown up is a pattern of requiring a major and a minor field, or a major and two minors, or two majors, and putting the requirements for these entirely in terms of numbers of semester hours. In mathematics, in 1957,§ the range for a major was between 12 and 40 semester hours, the median 18, and the average 20 hours. Very little attempt has been made to insure a uniform level of minimal competency in subject matter,‡ or to see that material needed in the high school is taught. It would be perfectly possible, for example, for a teacher whose certification major is mathematics never to have had a course in geometry after the tenth grade. Whatever control has been exerted has been by the individual colleges, and frequently the responsibility for subject-matter requirements has not been in the hands of the department concerned, even for a certification major. As for minors, often very little attention is paid by anybody to what a student takes to complete a minor.

Each state has made its own solution to the boundary-value problem. These differ widely, so that a teacher well prepared for one state may not be able to meet the neighboring state's requirements.

One can see obvious weaknesses in such a system. (One can also see major difficulties facing the reformer.) A less obvious weakness, that the NASDTEC members take very seriously, is that many teachers with minimum certifica-

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* Conant, *loc. cit.*

† Tabular Summary of Certification Requirements for School Personnel in the United States, 1959. National Education Association, Washington, D. C.

§ Conant, *loc. cit.*

‡ Several states give some sort of examination for certification, but very few states or cities test subject-matter proficiency.

tion majors, or who have ended up teaching in a minor, are not prepared to go on to a master's degree in the subject they are teaching. Since a master's degree has cash benefits, such teachers are apt to end up with a M.Ed., with no further subject matter training, or with as little as they can manage. This should be of special concern to us, because it is particularly in the tight fields of mathematics and the sciences that one finds people teaching with only a minor.

The NASDTEC study. The certification directors are working towards an "approved-program" approach. To quote their description,

"In this approach state departments of education and their advisory committees define guide lines for the preparation of teachers and leave to each college the responsibility of planning a program of courses and experiences. These programs are then submitted to the state department of education for approval. After approval is granted, an application for a certificate is considered on the basis of the applicant's having completed the approved program.

"Further, an effort is being made by state departments of education to accept out-of-state candidates by means of the 'approved program approach'. Therefore, the National Association of State Directors of Teacher Education and Certification at its June 1958 meeting agreed to support a national plan of reciprocity based upon accreditation of the preparation program by the National Commission for Accreditation of Teacher Education."

Clearly, the important question is, what will be the nature of these "guide lines" and how will they be determined?

Following Young's corollary to Parkinson's Law—the number of committees increases exponentially with time—the first step taken by the two organizations was to form a board of eight, five from NASDTEC, two from AAAS, including E. G. Begle of Yale and SMSG, and one from the Council of Chief State School Officers. John Mayor was appointed Study Director with Dr. William P. Viall, Chief, Bureau of Teacher Certification, New York State, as Associate Director and chairman of the board.

The first major step was the holding in April, 1960, of four regional meetings, covering all the states, to make recommendations for these guide lines. At each of these there were about 40 people. The meeting I attended was perhaps typical; there were present certification directors from the states of the region, eleven other educators in administrative positions including two state superintendents, and one principal; four high school teachers; one college president, and nine deans; four physical scientists, and five mathematicians, including two of the deans. Most of the actual work was done in small sessions of about ten people, each group taking a field and preparing a list of recommendations. The four sets of recommendations for each field are being circulated to a wider group for criticism. The revised versions will furnish the starting point for studies in each state.

Then finally in June, 1961, a national conference will be held

"with representation from each of the states, persons involved in the several other related studies, specialists in curriculum with particular attention to the fields of science and mathematics, and recognized national leaders in teacher education and in the sciences.

"The purposes of such a conference would be to review the studies and the recommendations

critically, to make suggestions for state and regional adaptation, to lay plans for the development of reciprocity agreements on a national basis, to inaugurate studies in other areas, and to bring to the attention of the public the purpose and progress of this work."

Following this conference, there will be a final report.

The nature of the mathematics recommendations. I can summarize rather quickly the reasons why there should be changes not only in the amount but in the nature of the training of the secondary teacher of mathematics. First, improvements in grades 1 through 8 through such projects as the SMSG program will mean better-prepared high school students, capable of more difficult work. Second, pressure from engineering schools and industry—and from some, though apparently not all, mathematicians—will push down into the high schools more and more work that is now usually done in college. Third, the current changes in the high school curriculum call for a different training. Thus the teacher of the next decade will be taking better-prepared students through new material well into college work. Many people would guess that in a decade 15% of our high school seniors will be taking calculus, a figure not unreasonable if compared with European experience.

In curriculum planning, mathematics appears to be ahead of the other sciences, and there seems to be general agreement on the nature of the changes required. Among the groups in mathematics that have considered the associated problem of teacher training, there is also agreement. With the benefit of all this work, it is not surprising that the reports of the regional groups are very much alike. I will try to construct a sort of average.

The center of the teacher's training should be three year courses. These are: (1) Calculus, or analytic geometry and calculus, with most of the emphasis on calculus, stressing basic concepts and intuitive ideas as well as or rather than manipulative techniques. (2) Modern algebra—the structure of the number system, groups, rings, fields, polynomials, matrices. (3) Geometry—a critical study of the foundations of geometry. Several other topics, such as projective or non-Euclidean geometry or analytic geometry by vectors.

Note that a poorly prepared student may very well have to take a year's work to begin this sequence, work of a nature that the groups did not believe should count as part of a teacher's training. (Indeed, several groups stated explicitly their dissatisfaction with such courses as solid geometry, college algebra, trigonometry, theory of equations, advanced Euclidean or college geometry, and mathematics of finance, courses which have been a large part of many teachers' majors.)

Beyond these three courses the student should have a year's work in courses such as probability and statistics, set theory and elementary logic, or additional work in one of the three main divisions.

I think there was probably general agreement that a minor should include the three main courses.

Fifth-year programs should be advanced courses of a somewhat similar na-

ture, and should prepare the teacher to teach the analytic geometry and calculus for the Advanced Placement Program of the College Entrance Examination Board. If the teacher has only a minor, the fifth year should complete the major program.

In most colleges and universities, the certification to the state that a student has completed the necessary work for a certificate and is competent to be in teaching is in the hands of the Education School or Department. So is the supervision of the student's practice teaching. These, of course, are purely internal matters for the college or university, and not determined by the state. There appeared general agreement that the subject-matter department play more of a part in both of these activities, taking at least the responsibility of attesting the student's mathematical competency.

Some personal observations. The four-year program is weaker than I had hoped for. This is, however, not the result of compromise between educators bent on keeping school standards low and high-principled university scholars nobly fighting for scholarship. The NASDTEC members seemed to me to be as anxious to improve the training of the teacher as I, and to be much better informed as to the obstacles, and perfectly willing to fight through them. If the University people had said that a year of Banach algebra was vital for the teacher, and explained what was involved, I felt the directors would have looked worried, and then gone out and worked to get it. It was the facts of the situation that forced us into the above recommendations as a minimal program, not merely a desire for agreement.

I should like to emphasize that this is a minimal program. I now believe that any competent department of mathematics, in almost all states, can draw up its own program for the teacher, and if it really is sensible, the state department of education will approve it.

I should like to emphasize also that this is a program for a particular point in history. In another ten years, if all goes well, the prospective teacher will enter college with a good bit stronger background in mathematics, and this program will be unduly weak.

The fifth-year program still has many problems. There seems almost universal agreement that it should be a year of further subject-matter study. Yet equating advanced work to a master's degree of the traditional sort, as I have said, almost forces most teachers into an education degree. There seems to me to be an urgent need of special master's degrees aimed at the teacher. As I interpret the position of the curriculum directors on requiring the degree, the degree means that someone besides a local school board has approved a program of study and certified satisfactory completion. Without the degree, entirely too much power is given to local school boards to judge the qualifications of the teacher.

Conclusion. The NASDTEC study is the first occasion that I know for mathematicians to influence directly the policy of state departments of educa-

tion and educational administrators on a nationwide scale. The state studies of the next year offer a chance, not only to help determine policy on an important question, but to gain a voice in educational policy at the highest state level. I believe that the opportunity is opening as a consequence of the evidence we have given in recent years of deep interest in the problems of the high school and of our willingness to work cooperatively with the secondary school people. Such a chance should be seized. In my opinion, the strongest mathematics departments in each state, the ones that should be providing guidance and leadership, should write their state certification officer, and express their willingness to help in the state study. I think you will be welcomed.

WHAT MATHEMATICS IS TAUGHT IN EUROPEAN SCHOOLS

HOWARD F. FEHR, Teachers College, Columbia University

The Organization of European Economic Cooperation has recently sponsored two events of importance for mathematics instruction in the western world. The first of these was an International Seminar in mathematics held at Royaumont, France, from November 21 to December 5, 1959, which was attended by fifty-one mathematicians and teachers from twenty-one countries of Europe and America. The other is a Survey of European Mathematics Education in which the Mathematical Association of America participated with funds supplied by the National Science Foundation. The complete reports of both these events will be published in the near future. The following article, indicative of some of the findings, is made from a preliminary examination of the Survey Questionnaires.

In making any comparison of the subject matter taught in European schools with that taught in the schools of the United States, one must consider the structure of the European schools. Generally, the school organization, the purposes of instruction, and selection or guidance of pupils is quite distinct from country to country in Europe and in all cases quite different from that held in the United States. In several countries of Europe, children begin formal schooling at ages 5, 6, or 7 years, and the elementary and secondary school program lasts for 11, 12, or 13 years of study. In order to make a description as valid as possible, I shall speak therefore only of that mathematics taught from ages 6 to 17 years.

At another place I have reported on the mathematics taught to the youth of the world, ages 6 to 15 years.* Here, then, I shall speak only briefly on arithmetic instruction. The arithmetic taught in the United States is exactly the same arithmetic as taught in Europe to children 6 to 12 years of age. The grade placement of the subject matter, however, varies considerably and generally, the subject matter covered in the books and classes in the United States is

* The mathematics education of youth—a comparative study, Enseignement Math., vol. 5, 1959, pp. 61–78.

placed about one-third academic year later than the average placement for all European countries. This means that some countries delay longer than the United States in introducing topics, *e.g.*, common fractions, while they also introduce other topics a year earlier than we do. The variation in placement is indicated in the following three problems:

(1) 68+25 is taught in:

1 country at age 6		4 countries at grade 1
13 countries at age 7	or	15 countries at grade 2
5 countries at age 8		1 country at grade 3
1 country at age 9		

Thus this problem is taught on the average at age 7½ years in grade 2, which is exactly the same time and place of instruction in the United States.

(2) Add: 784.927 is taught in:

27.384
63.673
591.595

3 countries at age 8		1 country at grade 2
6 countries at age 9	or	4 countries at grade 3
8 countries at age 10		9 countries at grade 4
3 countries at age 11		4 countries at grade 5
		2 countries at grade 6

There is great divergence in the placement of this topic. The average placement is in grade 4 toward the age 10 years. In the United States it is generally placed in grade 6 at age 11½ years. The use of the metric system may be one factor in establishing the earlier placement of this subject in Continental Europe.

(3) $2\frac{3}{4}+5\frac{7}{12}$ is taught in:

9 countries at age 10		6 countries in grade 5
5 countries at age 11	or	10 countries in grade 6
5 countries at age 12		3 countries in grade 7

The average placement is at the end of the 5th grade or beginning of the 6th grade at age 11½ years, while in the United States it occurs in the middle of the 5th grade at age 10½ years. The early placement of fractions in England and the United States may be accounted for in the use of our systems of measures, most of which are non-metric.

The teaching of arithmetic as a meaningful mathematical structure is more prevalent in the United States than in any other country of the world. This accounts in part for the delay in placement of some topics. Generally in Europe, while there is a trend to shift to meaningful teaching, the teaching still consists of rote, repetitive, manipulative drill on facts and processes. However, there is a more powerful factor at work in European schools to make for earlier placement of topics. This is the segregation of pupils into intellectual streams at the end of the 4th, 5th, or 6th grade or at ages ten to twelve years. The segregation is usually based on an examination consisting of two subjects—the native language and arithmetic. The whole teaching of arithmetic in grades 1 to 5 or 6 is aimed at preparing those pupils, who can, to pass this examination. For Europe

as a whole, almost one-half (50%) of the children do not take the examinations because (a) they, by class, relegate themselves to a lower working level than the professions which demand higher study and greater means of financing, or (b) they know that it would be totally impossible for them to pass the examination.

Of those pupils who take the examination, about 60 to 70 per cent pass it. Thus in Europe of all children age 12 years, only 30 out of every 100 enter the gymnasium, lycee, gymnas, grammar school, or other college preparatory school. The seventy per cent of the children who are barred from such study, continue in other schools until age 14 or 15 years and then enter either into the world of work or into vocational schools for further training. The arithmetic taught in European schools to children age 6 to 12 years is geared to the learning capacity of the upper 30 to 50 per cent in cognitive intelligence. The lower half learn very little. The caliber of instruction in the United States is known, but nevertheless, in the writer's judgment, it would be a great step backward if the United States were to adopt any of the present European program in place of the program evolving in our own schools. There I am sure we shall develop administrative and curriculum procedures to take care of those highly capable children we have so long neglected, and at the same time continue to teach arithmetic to all the children, adapting the instruction to their capacity and rate of learning.

Turning to the high school (grades 7 through 12), we will here consider only the mathematics taught in college preparatory programs. We have heard of the fine teaching and the thoroughness in learning of the subjects taught in European schools. Generally this is true, and it is primarily due to the excellent mathematical and pedagogical training of the teachers in these schools, as well as to the limitation for entrance to pupils with high abilities to learn. How would we, in the United States, enjoy teaching only the upper 30 per cent of intelligence in grades 7 and 8, the upper 15 per cent of intelligence in grades 9 and 10, and the upper 8 per cent of intelligence in grade 12! For these percentages show the average "student mortality rate" of secondary-school students in Europe.

Yet the subject matter covered is not vastly different from that in our schools and in no case is it modern in the sense of the fine materials written by the School Mathematics Study Group and the University of Illinois School Mathematics Committee, or the program advocated by the Commission on Mathematics. Let us look first at what is not taught in European schools.

1. There is no use of sets, set notations, or set language. Unions, intersections, Cartesian products, function as a mapping or a restricted set of ordered pairs, is nowhere present in the program. Function in all the European schools is merely $f(x)$ where x is a numerical variable.

2. The study of permutations, combinations, and probability, has been dropped in most schools as too difficult and designated as university study.

Statistics, as an organized study, both descriptive and inferential is not generally taught in the secondary school. There are three experiments in the teaching of this subject (Netherlands, Switzerland, and Sweden), but it is not a part of the official program. In England it is an optional subject in the sixth form which is really junior-college-level instruction.

3. The trigonometry of real numbers is nowhere to be found in the secondary school. The treatment of trigonometry is entirely based on angles (including the radian measure of angles). The wrapping function is completely unknown.

4. Formalism and axiomatic development are not stressed. To be sure, in the teaching of geometry there are definitions, axioms, theorems and proof, but not in so rigorous a manner as we use in our teaching. Proof in algebra (*i.e.*, using the usual laws of a group, ring, or field to prove further properties) is never mentioned. The study of formal algebra is strictly a university study, and the secondary school algebra can still be classified as essentially that of Hall and Knight.

5. The theory of determinants (except for 2nd- and 3rd-order determinants) and matrices is not taught in European secondary schools.

6. Of course complex numbers are taught and used, in the form $a+bi$, and in trigonometry in the form $(\cos\theta+i\sin\theta)$ but never in a formalized structure as ordered pairs of real numbers.

7. Projective geometry as a separate subject, (*i.e.*, more than a mention of a few properties, such as dualism) and non-Euclidean geometries do not exist in the instructional program. The study of groups is reserved for special students in the 13th year of study.

Turning to what is taught in Europe and generally not taught in the United States, we see some striking contrasts. A few of these differences demand careful study by American educators, for they give promise of valuable additions to our instruction in mathematics.

1. The first of these is the teaching of the calculus. This is a universal subject in European secondary school instruction and includes the differentiation and integration of the usual functions, with applications. The treatment is informal, with geometrical and physical explanations of the theory, and no attempt is made to give rigorous delta-epsilon proofs of theorems. The students learn how to do and how to use the calculus and they apply it to their study of physics. This study of calculus is to the formal calculus taught to them later in the university as our study of intuitive geometry of junior high school is to the study of deductive geometry in the high school. It gives a basic wide knowledge and experience upon which a formal mathematical system can be constructed.

2. Most European countries continue to teach descriptive geometry as a separate discipline (only The Netherlands has recently dropped this subject). This instruction is more than mechanical drawing, for it involves not only the making of drawings, but the theory of projections, representations, parallelism, perpendicularity, rotations, intersections and development of surfaces. This subject is never taught in the secondary school in the United States and is

found under the title of Engineering Drawing in some of our colleges and universities.

3. Cosmography is a subject quite common in continental European secondary schools. In this course the teaching of trigonometry is extended to the solution of spherical triangles. The celestial sphere and its coordinate systems are studied, and problems are solved in the determination of time, latitude, longitude, and orbits of planets. Recent interest in outer space, satellites, and missiles suggest that this elementary study of cosmography may be a good basis for the more extended and deeper mathematical treatment of space and orbit problems to be studied later in the university.

4. A treatment of Geometry as the set of transformations of a plane into itself (or space into self), called motion geometry (or Bewegungsgéometrie) is gaining considerable attention in Germany where 70 per cent of the gymnasium classes are using this approach to the subject. Here rotations, translations, and reflections (Spiegelungen) are developed intuitively, then the properties of these transformations are used in developing the usual geometry of the plane. Shearing, stretching, and contracting lead to the study of similarity and the introduction of vectors. This program is in harmony with the geometry program proposed by Felix Klein and called the Erlanger program, but it also leads to a better insight into the properties that distinguish one type of geometry from another.

5. I also mention the fundamental ideas of the theory of numbers. While in Europe the arithmetic of the elementary school is very mechanical, the attempt to mathematicize this arithmetic and uncover some of its basic properties is done in the secondary school, after the pupil has studied some algebra. Thus, division $a \div b$ becomes $a = bq + r$, where $r < b$, and from this understanding the Euclidean algorism for finding the greatest common divisor, theorems on divisibility, least common multiples, prime numbers, unique factorization, and the solution of simple Diophantine equations are developed.

6. Lastly, the properties of the conics, developed either synthetically or analytically, occupy a large part of the senior secondary school instruction. In our country this mathematics is fast disappearing, even at the university level, for it represents a treatment that can be greatly simplified by the use of the calculus. Further, most of the properties studied have little to do with either practical applications or the subsequent study of mathematics. However, since the synthetic study of the properties of conics requires a type of mathematical insight and patience for complex thinking, it makes this study an excellent sieve in the selection of university students who will major in mathematics.

Thus we see that, in large measure, Europe teaches the same mathematics we do, albeit with greater depth and complexity, and usually to pupils at a slightly younger age than ours, because their pupils are more highly selected. They also teach mathematics we do not include in our program, not because we feel this mathematics is too difficult, but because through the traditional and cultural patterns we have acquired, we feel it is either not of value or of need to our pupils. More important, however, is the fact that whereas the

European program is still relatively static there are emerging in the United States a number of mathematics programs that include present-day concepts and symbolism, that are in line with technological and scientific needs of our country, and which are capable of study and grasp by a much larger percentage of the youth than that which could successfully pursue the present classical program of European schools.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1431. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Advanced Development*

If $a_{n+1}=(1+a_n a_{n-1})/a_{n-2}$ and $a_1=a_2=a_3=1$, show that a_n is an integer.

E 1432. *Proposed by O. Lowenschuss and A. Rosenfeld, Budd Lewyt Electronics, Inc., Long Island City, N. Y.*

How many distinct products appear in the multiplication table through $N\times N$?

E 1433. *Proposed by Alexander Oppenheim, University of Malaya*

Let P be a point in the interior of a triangle and let the distances of P from the vertices of the triangle be x, y, z and from the sides of the triangle be p, q, r. Show that $xyz \geqq (q+r)(r+p)(p+q)$.

E 1434. *Proposed by Anatole Beck, University of Wisconsin*

Show that every open set in the plane can be represented as a disjoint union of closed straight line segments.

E 1435. *Proposed by J. F. Leetch, Ohio State University*

Find the limit superior and the limit inferior for each of the following sequences of sets of real numbers:

(a) $\{A_n\}_{n=1}^{\infty}$ where A_n is the set of integers mod n,

(b) $\{A_n\}_{n=1}^{\infty}$ where $A_n=\{a/n \mid a\in J \text{ (integers)}\}$.

SOLUTIONS

The Derivatives of $1/(e^u-1)$

E 1401 [1960, 182]. *Proposed by Emil Grosswald, Institute for Advanced Study*

Prove that for every integer $k \geqq 1$,

$$\frac{d^{k-1}}{du^{k-1}}\left(\frac{1}{e^u-1}\right) = (-1)^{k-1}\sum_{r=1}^{k}\frac{\gamma_k(r)}{(e^u-1)^r},$$

where the constants $\gamma_k(r)$ are all positive and satisfy

$$\gamma_k(r) = \sum_{\nu=0}^{r-1}(-1)^\nu\binom{r-1}{\nu}(r-\nu)^{k-1}.$$

In particular, show that $\gamma_k(k) = (k-1)!$.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since

$$\gamma_n(r) = \left[\frac{d^{n-1}}{dx^{n-1}}\left\{e^x(e^x-1)^{r-1}\right\}\right]_{x=0}, \qquad 1 \leqq r \leqq n,$$

we find that $\gamma_{n+1}(r) = r\gamma_n(r) + (r-1)\gamma_n(r-1)$. Since $e^x(e^x-1)^{r-1} = (e^x-1)^r - (e^x-1)^{r-1}$, we see that $\gamma_n(n) = (n-1)!$ and $\gamma_n(n+1) = 0$. The desired formula is true for $k=1$. Suppose now the formula is true for $k = 1, 2, \cdots, n$. Then

$$\begin{aligned}(-1)^n\sum_{r=1}^{n+1}\frac{\gamma_{n+1}(r)}{(e^u-1)^r} &= (-1)^n\left[\sum_{r=1}^{n}\frac{r\gamma_n(r)}{(e^u-1)^r} + \sum_{r=1}^{n}\frac{r\gamma_n(r)}{(e^u-1)^{r+1}}\right] \\ &= (-1)^n\sum_{r=1}^{n}\frac{e^u r\gamma_n(r)}{(e^u-1)^{r+1}} \\ &= \frac{d}{du}\left[(-1)^{n-1}\sum_{r=1}^{n}\frac{\gamma_n(r)}{(e^u-1)^r}\right] = \frac{d^n}{du^n}\left(\frac{1}{e^u-1}\right).\end{aligned}$$

Thus, by mathematical induction, the formula is true for integer $n \geqq 1$.

Also solved by A. N. Aheart, D. W. Bailey, Leonard Carlitz, G. DiAntonio, Jane Evans, S. H. Greene, A. F. Kaupe, Jr., M. S. Klamkin, Donald Knuth, D. C. B. Marsh, C. F. Pinzka, B. E. Rhoades, Herbert Ruderfer, Vencil Skarda, and the proposer.

DiAntonio pointed out that this problem is essentially Prob. 8 on p. 87 of Gibson's *Advanced Calculus*. Pinzka pointed out that the concluding particular result follows from Problem 4183 [1947, 235].

An Inequality Connecting the Inradius and Exradii of a Tetrahedron

E 1402 [1960, 182]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let r denote the radius of the inscribed sphere of a tetrahedron T and let r_i $(i=1, 2, 3, 4)$ denote the radii of the exspheres of T which touch one face of T and the other three faces of T produced. Show that $\sum_{i=1}^{4} r_i \geqq 8r$, with equality if and only if T is isosceles.

Solution by Joe Lipman, Harvard University. If $3S=\sum_{i=1}^{4} a_i$, where a_i is the area of the ith face, then $r_i=V/(S-2a_i/3)$ and $r=V/S$, where V is the volume of the tetrahedron. This gives $\sum_{i=1}^{4}(1/r_i)=2/r$. But, by elementary algebra, $\sum_{i=1}^{4}(1/r_i)\sum_{i=1}^{4} r_i \geqq 16$, with equality if and only if the r_i, that is the a_i, are equal. The conclusion follows.

Also solved by John Burr, J. W. Clawson, M. S. Klamkin, D. C. B. Marsh, and the proposer.

A Property of a Polynomial

E 1403 [1960, 182]. *Proposed by T. J. Rivlin, IBM Corp., Yorktown Heights, N. Y.*

Let $p(x)=a_0+a_1x+\cdots+a_nx^n$, with the sequence of coefficients a_3, a_5, a_7, $\cdots$, a_m ($m=n$, n odd; $m=n-1$, n even) either all nonnegative or all nonpositive. Show that:

(a) there exists a number y_n such that $\{p(1)-p(-1)\}/2=p'(y_n)$,
(b) if n is odd and >1, $|y_n| \leqq (1/n)^{1/(n-1)}$,
(c) if n is even and >2, $|y_n| \leqq \{1/(n-1)\}^{1/(n-2)}$,
(d) the bounds on $|y_n|$ are the best possible,
(e) when $n=1$, 2 we may take $y_n=0$.

Solution by D. C. B. Marsh, Colorado School of Mines. Write

$$p'(x) - \{p(1) - p(-1)\}/2 \equiv \phi(x) \equiv f(x) + xg(x),$$

where, for $n \geqq 3$,

$$f(x) = \sum_{j=1}^{(m-1)/2} \{(2j+1)x^{2j} - 1\}a_{2j+1}, \qquad g(x) = \sum_{j=1}^{[n/2]} 2ja_{2j}x^{2j-2},$$

and [ ] denotes the greatest integer function. For $m>1$, let $x^*=(1/m)^{1/(m-1)}$. It is easy to show that, for $j \leqq (m-1)/2$, $(2j+1)(x^*)^{2j} \geqq 1$, so that $f(x^*)$, $f(-x^*)$, and $-f(0)$ have the same sign as the numbers of the set a_3, a_5, $\cdots$, a_m. Now $g(x)$ is an even function so that $x^*g(x^*)$ and $-x^*g(-x^*)$ have opposite signs, one being the same as the sign of the numbers a_3, a_5, $\cdots$, a_m. Thus one of $\phi(x^*)$ and $\phi(-x^*)$ has the same sign as the numbers a_3, a_5, $\cdots$, a_m, whereas $\phi(0)$ has the opposite sign. With $\phi(x)$ continuous there must therefore exist at least one value of x (say y_n) between x^* and $-x^*$ for which $\phi(y_n)=0$, and thus a value for which $p'(y_n)=\{p(1)-p(-1)\}/2$, proving (a), (b), and (c). For odd $n>1$, when $a_j=0$ ($j \leqq n-1$) and $a_n=1$, $\phi(x)=0$ is satisfied by only $x=\pm x^*$ in the interval $(-x^*, x^*)$. For even $n>2$, when $a_j=0$ ($j<n-2$), $a_{n-2}=1$, $a_{n-1}=2\{1/(n-1)\}^{(n-3)/(n-2)}$, $a_n=\{(2-n)/n\}(n-1)^{2/(n-2)}$, we have $\phi(x)=0$ is satisfied only for $x=\pm x^*$ in the interval $(-x^*, x^*)$, proving (d). For $n=1$, the equation becomes $a_1=a_1$, satisfied identically, so we *may* choose $y_n=0$, while for $n=2$, one has $a_1=a_1+2a_2y_n$, permitting $y_n=0$. Thence (e).

Also solved by Jane Evans, S. H. Greene, and the proposer.

The proposer wondered if there is some simple bound for y_n when the coefficients of the polynomial are completely arbitrary.

A Property of the Two-Dimensional Point Lattice

E 1404 [1960, 183]. *Proposed by H. N. Ward, Harvard University*

(a) In Euclidean 2-dimensional space R^2, let L be the lattice of points with integral coordinates, and T any nondegenerate triangle (including its boundary) whose vertices belong to L. Let T' be any translation of $-T(=\{-x|x\in T\})$ which has members of L as vertices; that is, $T'=p-T$, where $p\in L$. Show that if $T\cap T'\neq\phi$, then $T\cap T'$ contains a member of L.

(b) Show that an analogous statement in Euclidean n-dimensional space R^n, $n>2$, where simplexes are used for triangles, fails.

Solution by the proposer. Let R^n be Euclidean n-space; let $e_1, \cdots, e_n$ be a fixed basis; and let L^n be all points $p=a_1e_1+\cdots+a_ne_n$ for which the a_i are integers. Then L^n is the integral lattice. For any two subsets A and B of R^n let $A+B$ be all $a+b$ with a in A, b in B. Let A^m stand for $A+\cdots+A$ with m summands. Let A^* be $A\cap L^n$. For any scalar r, let rA be all ra for a in A.

Let A be a convex set in R^n. Then for any positive integer m, $A^m=mA$. For an element of A^m is $p_1+\cdots+p_m$ where each p_i is in A. But then $(1/m)p_1+(1/m)p_2+\cdots+(1/m)p_m=p$ is in A from the convexity, so that $p_1+\cdots+p_m=mp$ is in mA. Then $A^m\subseteq mA$. But if p is in A, $mp=p+\cdots+p$ (m summands) is in A^m, so $mA\subseteq A^m$; that is then, $A^m=mA$.

Now suppose A is convex. Suppose also A is the union of a family $\{A_i\}$ (with a given index set I) of convex sets such that for each A_i one has that $(A_i^*)^m=((A_i)^m)^*$ for all m. Then in fact $(A^*)^m=(A^m)^*$. For, $A^m=mA=\cup mA_i$, and since the A_i are convex, this last is $\cup(A_i)^m$. Intersecting the sets with L^n, one has $(A^m)^*=\cup((A_i)^m)^*=\cup(A_i^*)^m$. Here all unions are over the set I. Say now that $p_1+\cdots+p_m$ is in $(\cup A_i^*)^m$. Suppose p_j is given to be in A_{ij}^*. Then, since A is convex and each p_j is in A, one has that $(1/m)p_1+\cdots+(1/m)p_m$ is in A. Then this last element is in A_i for some i, so that $p_1+\cdots+p_m$ is in $mA_i=(A_i)^m$. Since it is also in L^n because each p_j is, it is then in $((A_i)^m)^*=(A_i^*)^m$. Then $p_1+\cdots+p_m$ is in $\cup(A_i^*)^m$. This last union is a subset of $(\cup A_i^*)^m$, which is $(A^*)^m$. So one has $(\cup A_i^*)^m\subseteq\cup(A_i^*)^m\subseteq(\cup A_i^*)^m=(A^*)^m$. Since the second set in the string is $(A^m)^*$, one has the desired result.

In R^2 let C be a closed convex polygon whose vertices are in L^2. Then (calling such a polygon a lattice polygon) one has that C is a union of lattice triangles $\{T_i\}$ (which are taken to be closed). If one can show that for a lattice triangle T one has $(T^*)^m=(T^m)^*$, the preceding will give $(C^*)^m=(C^m)^*$. Let T be a lattice triangle whose vertices are $a_ie_1+b_ie_2$, $i=1, 2, 3$. Let the parallelogram 0, e_1, e_2, e_1+e_2 be given unit area. Then the area of T is $|D/2|$, where D is the determinant whose ith row is a_i, b_i, 1. The area is a half-integer. Then if one can show that any lattice triangle is the union of lattice triangles of area $\frac{1}{2}$, and that

for any lattice triangle of area $\frac{1}{2}$ one has the relation $(T^*)^m=(T^m)^*$, then again the earlier result will give the desired conclusion.

Let (x, y) be written for xe_1+ye_2. Say T is a lattice triangle of area $\frac{1}{2}$. There is no loss in generality in taking one vertex as 0. Say (a, b) and (c, d) are the other vertices, ordered to give $ad-bc=1$, the value of $|D|$. Suppose (x, y) is a lattice point of T^m. Then $(x/m, y/m)$ is in T. Then $(x/m, y/m)=r(a, b)+s(c, d)+t0$, where r, s, and t are nonnegative and $r+s+t=1$. That is, $x/m=ra+sc$, $y/m=rb+sd$. Solving, one has $r=dx/m-cy/m$, $s=ay/m-bx/m$; that is, $r=u/m$, $s=v/m$, where the u, v must be nonnegative integers for which $u+v$ is at most m. Then $(x, y)=u(a, b)+v(c, d)+(m-u-v)0$, which is a point of $(T^*)^m$. So $(T^m)^*\subseteq(T^*)^m$. Since the reverse is always true, one has $(T^m)^*=(T^*)^m$.

Now let T be a lattice triangle of area equal to or greater than $\frac{1}{2}$; one must show that it is the union of lattice triangles of area $\frac{1}{2}$. One can prove this inductively since the area is always a half-integer. In fact, it is sufficient to show that such a T contains a point of L^2 other than its vertices; for using such an additional point as a vertex, one can decompose T into two or three lattice triangles according as the point is on an edge or in the interior of T. Each such triangle has area strictly less than that of T, and the induction hypothesis works; the statement is true at area $\frac{1}{2}$.

Let the area be S; one may take $S\geqq 1$. Assume one vertex is 0, and say p and q are the others. The parallelogram whose vertices are $p+q$, $p-q$, $-p+q$, $-p-q$ has area $8S\geqq 8$. Call P the parallelogram and form $\sqrt{\frac{1}{2}}P$, whose area is $\geqq 4$. Then by Minkowski's theorem on compact convex sets symmetric about 0 with area greater than or equal to 4, $\sqrt{\frac{1}{2}}P$ contains a lattice point other than 0. Then P contains a lattice point other than 0, which, because of the reducing factor, is also different from any of the vertices. Because P is a union of translations by members of L^2 of T and $-T$, T has a point of L^2 in it other than its vertices, as needed; for, in fact, these translated triangles can be taken as having vertices among the points 0 and the four vertices of P.

One immediate conclusion is a slight generalization of the result asked for in E 1404: let C be any convex lattice polygon and suppose for a given p in L^2 that C meets $p+(-C)$. Then, in fact, there is a member of L^2 in the intersection. For, one must have points a and b in C for which $a=p-b$. Then $p=a+b$; so p is in $(C^2)^*$. But then p is in $(C^4)^2$; so that there are points c and d of L^2 in C^* with $p=c+d$. But then the lattice point $c=p-d$ is a point of L^2 common to C and $p+(-C)$ as wished.

Another corollary is a sort of figurate number result. Let C be a convex lattice polygon; let C_e denote the set formed by the edges, and C_i the interior. Let $S(C)$ denote the area; and for any set A let $N(A)$ be the number of points of L^2 in A. One has the area formula $S(C)=N(C_i)+N(C_e)/2-1$. But for an integer m, $S(C^m)=m^2S(C)$, and $N((C^m)_e)=mN(C_e)$. Then one has that the number of lattice points in $(C^*)^m$ is a polynomial in m of the second degree; for this number is $N((C^*)^m)=N((C^m)^*)=N(C^m)=N((C^m)_e)+N((C^m)_i)=mN(C_e)$

$+S(C^m)-N((C^m)_e)/2+1=mN(C_e)+m^2S(C)-mN(C_e)/2+1$; that is, one has that $N((C^*)^m)=m^2S(C)+mN(C_e)/2+1$.

The Series $1+22+333+\cdots$

E 1405 [1960, 183]. *Proposed by L. L. Garner, University of North Carolina*

Find (a) an expression for the general term, (b) the sum of the first n terms of the series $1+22+333+\cdots$.

Solution by W. B. Carver, Cornell University. The problem is indefinite. The three given terms are the first three terms of infinitely many different series, as, for instance, the series whose general term is $u_k=145k^2-414k+270$, for which the sum of n terms would be $S_n=(290n^3-807n^2+523n)/6$.

However, I *guess* that the proposer intended that the general term of the series should be $u_k=k(10^k-1)/9$. In that case the sum of n terms would be

$$S_n=\sum_{k=1}^{n}(k\cdot 10^k-k)/9.$$

Then

$$9S_n=(1\cdot 10+2\cdot 10^2+3\cdot 10^3+\cdots+n\cdot 10^n)-(1+2+3+\cdots+n).$$

The series in the first parentheses is a combination arithmetic-geometric series which can be summed by the same method as that used for summing the geometric series. Let

$$\begin{aligned} P &= 1\cdot 10+2\cdot 10^2+3\cdot 10^3+\cdots+n\cdot 10^n,\\ 10P &= \qquad 1\cdot 10^2+2\cdot 10^3+\cdots+(n+1)\cdot 10^n+10^{n+1}. \end{aligned}$$

Hence, subtracting,

$$\begin{aligned} 9P &= -(10+10^2+10^3+\cdots+10^n)+n\cdot 10^{n+1}\\ &= -10(10^n-1)/9+n\cdot 10^{n+1}, \end{aligned}$$

and

$$\begin{aligned} S_n &= -10(10^n-1)/729+n\cdot 10^{n+1}/81-n(n+1)/18\\ &= [(9n-1)\cdot 10^{n+1}+10]/729-n(n+1)/18. \end{aligned}$$

It is an interesting exercise to show directly that this expression for S_n gives an *integer* for all integral values of n.

Also solved by A. N. Aheart, R. H. Anglin, D. W. Bailey, J. W. Baldwin, Edward Barbeau, Merrill Barnebey, Alan Beal, W. R. Becker, Roy Bertoldo, A. P. Boblétt, Julian Braun, D. A. Breault, A. M. Broshi, Brother Christopher Beaulieu, J. L. Brown, Jr., A. W. Brunson, R. G. Buschman, J. W. Clawson, F. B. Correia, A. E. Crofts, Jr., Thomas Curry, T. M. Delmer and L. F. Klosinski (jointly), G. DiAntonio, E. S. Eby, Walter Ehrenpreis, L. F. Epstein, Jane Evans, Joyce Farrell, A. E. Fleck, Charles Franti, Michael Goldberg, L. D. Goldstone, H. W. Gould, Mrs. R. H. Gramann, S. H. Greene, Emil Grosswald, Margaret Herzog, H. W. Hickey, Geoffrey

Horrocks, A. R. Hyde, J. M. Kingston, M. S. Klamkin, Donald Knuth, Harry Langman, Joe Lipman, Stephen Lowe, D. C. B. Marsh, J. W. Mettler, N. G. Mouck, Jr, C. S. Ogilvy, M. J. Pascual, R. Q. Petersen, D. J. Peterson, J. L. Pietenpol, C. F. Pinzka, Thomas Porsching, J. H. Rascoff, B. E. Rhoades, D. A. Robinson, J. V. Ryff, R. S. Sabharwal, Barbara Sakitt, E. M. Scheuer, Y. P. Sharma, C. M. Sidlo, Stephen Stein, M. F. Stilwell, R. P. Tapscott, R. C. Weger, T. T. West, R. J. Wisner, C. C. Yalavigi, Walter Zayachkowski, and the proposer.

Editorial Note. A couple of other elementary methods of obtaining $T_n(x) = \sum_{k=1}^{n} kx^k$, $x \neq 1$, are:

(1)
$$\begin{aligned} T_n(x) &= x + 2x^2 + 3x^3 + \cdots + nx^n \\ &= (x + x^2 + x^3 + \cdots + x^n) + (x^2 + x^3 + \cdots + x^n) + \cdots + x^n \\ &= x(x^n - 1)/(x - 1) + x^2(x^{n-1} - 1)/(x - 1) + \cdots + x^n(x - 1)/(x - 1) \\ &= [nx^{n+1} - (x + x^2 + \cdots + x^n)]/(x - 1) \\ &= [nx^{n+1}(x - 1) - x(x^n - 1)]/(x - 1)^2 \\ &= x[nx^{n+1} - (n + 1)x^n + 1]/(x - 1)^2. \end{aligned}$$

(2)
$$\begin{aligned} T_n(x) &= x(d/dx)\left(\sum_{k=1}^{n} x^k\right) \\ &= x(d/dx)[(x^{n+1} - x)/(x - 1)] \\ &= x[nx^{n+1} - (n + 1)x^n + 1]/(x - 1)^2. \end{aligned}$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4923. *Proposed by D. J. Newman, Brown University*

For any set of real numbers, S, we define its "divisor set," $\hat{S}$, to be the set of all numbers in (0, 1) for which some integral multiple lies in S. Are there sets S of arbitrarily large measure for which $\hat{S}$ has arbitrarily small measure?

4924. *Proposed by J. H. Blau, Antioch College and Stanford University*

Let P denote a partition of $[0, 1]$ into n disjoint measurable sets E_i. Denote $\sup m(E_i)$ by $|P|$, and let $x_i \in E_i$. For which functions F on $[0, 1]$ does the following pseudo-integral exist:

$$\lim_{|P| \to 0} \sum_{i=1}^{n} F(x_i)m(E_i) \text{ ?}$$

4925. *Proposed by C. Kacser, Clarendon Laboratory, Oxford, England*

There is a well-known result that

$$\sum_{n=0}^{\infty} (2n+1)P_n(x)Q_n(z) = \frac{1}{z-x},$$

where P_n and Q_n are the regular and irregular Legendre polynomials of order n. A suggested generalization is

$$\sum_{n=0}^{\infty} (2n+1)P_n(x)[Q_n(z)]^3 = \left[\log\frac{z-x}{z-1}\right]^2 \frac{1}{z-x},$$

at least in the limit as z approaches 1 from above. Prove or disprove.

4926. *Proposed by S. Berberian, State University of Iowa*

If $\mathfrak{H}$ is infinite-dimensional (say separable) Hilbert space, construct a densely defined linear transformation T for which the domain of the adjoint transformation T^* is the subspace $\{0\}$.

4927. *Proposed by David Gale, Brown University*

Suppose f is a continuous transformation of n-space into itself, and suppose there is a positive number λ such that $|f(x)-f(y)| \geqq \lambda|x-y|$ for all points x and y, where $|x-y|$ is the Euclidean distance. Prove that f maps onto all of n-space.

4928. *Proposed by C. C. Faith, Pennsylvania State University*

Show by a concrete example that there exists a normal equation $\phi(x)=0$ of degree n over a field F of characteristic 0 having two conjugates v and v' whose difference $v-v'$ has degree less than n over F, when $n=4$, or when $n=6$.

SOLUTIONS

A Defining Property of the Circle

4869 [1959, 816]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

A smooth centro-symmetric curve has the property that the centroid of any half-area which is formed by a chord through the center is equidistant from the center. Show that the curve is a circle.

Solution by Harley Flanders, University of California, Berkeley. Represent the curve in polar coordinates by $r=r(\theta)$, a periodic function with $r(\alpha+\pi)=r(\alpha)$. Assume r is differentiable. After multiplying r by a suitable constant, the condition on the centroid is

$$\left[\int_{\alpha}^{\alpha+\pi} r^3\cos\theta d\theta\right]^2 + \left[\int_{\alpha}^{\alpha+\pi} r^3\sin\theta d\theta\right]^2 = \left[\int_{\alpha}^{\alpha+\pi} r^2 d\theta\right]^2.$$

Differentiation with respect to α yields

$$\cos\alpha\int_{\alpha}^{\alpha+\pi} r^3\cos\theta d\theta+\sin\alpha\int_{\alpha}^{\alpha+\pi} r^3\sin\theta d\theta=0,$$

and a second differentiation,

$$-2r^3-\sin\alpha\int_{\alpha}^{\alpha+\pi} r^3\cos\theta d\theta+\cos\alpha\int_{\alpha}^{\alpha+\pi} r^3\sin\theta d\theta=0.$$

Another differentiation gives $r'=0$, whence r is a constant and the curve is a circle.

Also solved by the proposer.

Editorial Note. The proposer remarks that if the locus of the centroid were an ellipse instead of a circle, the same argument proves that the original curve must be a homothetic ellipse. He also conjectures that an analogous hypothesis regarding a surface in three-space will lead to a sphere.

A Locus Defined in Complex Coordinates

4870 [1959, 816]. *Proposed by D. S. Mitrinovitch, Belgrade, Yugoslavia*

If the variable z traces the circle $|z|=1$, find and describe the path of the point $w=(z^2-az)/(az-1)$, where the parameter a is a complex constant.

Solution by D. C. B. Marsh, Colorado School of Mines. Multiplication by $(az-1)\bar{z}$ enables one to write the transformation as $wa-w\bar{z}=z-a$, for points of the unit circle, $|z|=1$. Setting $z=x+yi$, $w=u+vi$, $a=A+Bi$, and separating reals and pure imaginaries, one finds

$$(u+1)x+vy=(u+1)A-vB,\qquad vx-(u-1)y=vA+(u+1)B,$$

assuming that a is truly complex (*i.e.*, $B\neq 0$). This linear system yields

$$x=A+\frac{2Bv}{u^2+v^2-1},\qquad y=-B-\frac{2B(u+1)}{u^2+v^2-1}.$$

so that $x^2+y^2=1$ becomes

$$(u^2+v^2-1)^2=\{A(u^2+v^2-1)+2Bv\}^2+B^2\{(u^2+v^2-1)+2(u+1)\}^2.$$

Establishing a T-axis orthogonal to the w-plane, this quartic may be considered as the orthgonal projection on the w-plane of the curve of intersection of the paraboloid $t+1=u^2+v^2$ and the oblique cone $t^2=(At+2Bv)^2+B^2(t+2u+2)^2$.

The quartic may also be constructed pointwise by assigning values to the parameter m, constructing the circles $u^2+v^2=1+2mB$ and $(u+1+mB)^2+(v+mA)^2=m^2$, and having their points of intersection as points on the quartic.

Also solved by A. J. Kokar.

A Fixed Point Theorem

4873 [1959, 817]. *Proposed by W. R. Utz, University of Missouri*

Let S be a connected topological space in which there is a relation $<$ which satisfies: (1) S has a least and a greatest element relative to $<$, (2) $<$ satisfies the trichotomy law, (3) $<$ is continuous. In other words, if $x<y$ in S, then there exist neighborhoods N_x and N_y of x and y, respectively, such that $z\in N_x$ implies $z<w$ for each $w\in N_y$.

Show that any continuous transformation of S into itself has a fixed point, and give an example of S which is not an arc.

Solution by Robert Speira, University of California, Berkeley. Let f be the continuous transformation. Assume that f has no fixed point. Let 0 and 1 be the least and greatest elements, respectively. Thus $f(0)\neq 0$ and $f(1)\neq 1$. Hence $0<f(0)$, and $f(1)<1$. Thus the sets $\{z: z<f(z)\}$ and $\{z: f(z)<z\}$ are both non-empty, and exhaust S, and are disjoint (using trichotomy and the assumption on f).

We now show both sets are open. Let $x\in\{z: f(z)<z\}$. Let $y=f(x)$. Then there exist open N_x, N_y satisfying the hypothesis of the problem. Now $f^{-1}[N_y]$ is an open set containing x. Thus $f^{-1}[N_y]\cap N_x$ is open and contains x; moreover it is included in $\{z: f(z)<z\}$. For if t is in the intersection, then $t\in N_x$ and $f(t)\in N_y$, hence $t<f(t)$. Thus $\{z: f(z)<z\}$ is open. A similar argument shows $\{z: z<f(z)\}$ is open. But then, S is partitioned into two open sets, contrary to the hypothesis of connectedness. Hence f must have a fixed point.

An example of such an S, not an arc, is the plane square, $0\leqq x\leqq 1$, $0\leqq y\leqq 1$. For the topology on S we take as base all vertical open segments. This topology contains the usual topology, as an open neighborhood can be written as the union of the open vertical segments contained within it. Define $(x_1, y_1)<(x_2, y_2)$ if and only if $(x_1<x_2)\vee(x_1=x_2\wedge y_1<y_2)$. This $<$ clearly satisfies the requirements, where the greatest and least elements are (1, 1) and (0, 0) respectively. This space is not homeomorphic to an arc, as any such mapping would be continuous in the usual sense since this topology contains the usual topology, and there is no one-to-one continuous function mapping the unit interval onto the unit square (Hobson, *Functions of a Real Variable*, Vol. I, §325).

Also solved by Ben Fitzpatrick, Jr., Harley Flanders, Gerald Leibowitz and Louis Hodes, P. S. Landweber, Elliott Mendelson, Jerrold Siegel, James V. Whittaker, and the proposer.

Fourier Coefficients

4874 [1959, 817]. *Proposed by D. J. Newman, Brown University*

Let c_n be a given sequence of constants such that $c_n=O(1/n)$, $n=1, 2, \cdots$. Show that there is a bounded function $f(x)$ with c_n as its positive Fourier coefficients, *i.e.*, such that

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = c_n.$$

Solution by the proposer. It is a theorem of F. Riesz that if (and only if)

$$\left|\sum_{n=1}^{N} c_n\lambda_n\right| \Big/ \int \left|\sum_{n=1}^{N} \phi_n\lambda_n\right| < M,$$

with M independent of N and the λ's, then the moment problem

$$\int \phi_n \cdot f = c_n, \qquad n = 1, 2, \cdots$$

has a solution with f a bounded function.

It is also a theorem of G. H. Hardy that

$$\sum_{n=1}^{N} |\lambda_n| / n \leqq M \int_0^{2\pi} \left|\sum_{n=1}^{N} \lambda_n e^{-inx}\right| dx,$$

where M is an absolute constant.

Since $2\pi c_n = O(1/n)$, an application of these two theorems gives the result.

The Coefficients of $x \cosh x/\sin x$

4875 [1959, 920]. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Prove the following conjecture: if

$$\frac{x \cosh x}{\sin x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{D_{2n}x^{2n}}{(2n)!},$$

then

$$D_{2n} = G_{2n} + \sum_{\substack{(p-1)\mid 2n \\ p\equiv 1(4)}} \frac{1}{p} + \sum_{\substack{p \leq 2n+1 \\ p\equiv 3(4)}} \frac{\epsilon_p}{p},$$

where the G_{2n} are integers and the ϵ_p are integers depending on p and n. Indeed, an explicit formula for ϵ_p can be obtained.

Solution by Leonard Carlitz, Duke University. Put

$$\frac{x \cosh x}{\sin x} = -\sum_{n=0}^{\infty} (-1)^n \frac{D_{2n}x^{2n}}{(2n)!}, \quad \frac{x}{\sin x} = \sum_{n=0}^{\infty} (-1)^n \frac{R_{2n}x^{2n}}{(2n)!}.$$

Then it is known (see Nörlund, *Differenzenrechnung*, pp. 27, 28) that

$$R_{2n} = 2(1 - 2^{2n-1})B_{2n}, \tag{1}$$

where B_{2n} denotes the Bernoulli number in the even suffix notation. It follows that

$$-D_{2n} = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n}{2r} R_{2r}. \tag{2}$$

Now from (1) and the Staudt-Clausen theorem we have, for $n>0$,

$$pR_{2n} \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid 2n) \\ 0 & (p-1 \nmid 2n); \end{cases}$$

thus (2) yields

$$pD_{2n} \equiv \sum_{\substack{r>0 \\ p-1 \mid 2r}} (-1)^{n-r} \binom{2n}{2r} \pmod{p}. \tag{3}$$

If $p \equiv 1 \pmod 4$, (3) becomes

$$pD_{2n} \equiv (-1)^n \sum_{\substack{r>0 \\ p-1 \mid 2r}} \binom{2n}{2r} \pmod{p}. \tag{4}$$

But by a formula of Hermite and Bachmann (for a short proof see *Mathematics Magazine*, Sept.-Oct. 1959, p. 7)

$$\sum_{0<r(p-1)<n} \binom{n}{r(p-1)} \equiv 0 \pmod{p}.$$

Thus (4) becomes

$$pD_{2n} \equiv \begin{cases} 1 \pmod{p} & (p-1 \mid 2n) \\ 0 & (p-1 \nmid 2n). \end{cases} \tag{5}$$

On the other hand, when $p \equiv 3 \pmod 4$, we get

$$pD_{2n} \sum_{r>0} (-1)^{n-r} \binom{2n}{r(p-1)} \pmod{p}.$$

Hence, if we put

$$\epsilon_p = \sum_{r>0} (-1)^{n-r} \binom{2n}{r(p-1)},$$

the conjectured result follows.

The formula for ϵ_p does not seem capable of simplification.

For similar results, see *Mathematics Magazine*, Mar.–April 1956, pp. 193–197; May–June 1957, problem 311; Mar.–April 1958, pp. 185–190.

Discontinuous Function with Partial Derivatives Everywhere

4876 [1959, 921]. *Proposed by Naoki Kimura, University of Washington*

If a real valued function $f(x, y)$ of two real variables possesses all of its partial derivatives $\partial^{m+n} f(x, y)/\partial x^m \partial y^n$ at every point, is it necessarily continuous?

Solution by John Burr, University of New England, Australia. The following

example shows that the function need not be continuous. The function $f(x, y) = \exp(-x^2/y^2 - y^2/x^2)$ $(xy \neq 0)$, $f(0, y) = f(x, 0) = 0$, is discontinuous at $(0, 0)$, since when $t \to 0$, $f(t, t) \to e^{-2} \neq f(0, 0)$. Suppose that it has been proved that a particular partial derivative $\phi(x, y)$ has the properties (i) $\phi(0, y) = \phi(x, 0) = 0$, (ii) if $xy \neq 0$ then $\phi(x, y) = R(x, y)f(x, y)$ where $R(x, y)$ is rational with denominator of the form $x^p y^q$. Then, by (i), $\phi_y(0, y) = \phi_x(x, 0) = 0$; by (ii) $\phi_x(0, y) = 0$ when $y \neq 0$, since when $x \to 0$, $f(x, y) \to 0$ more rapidly than any power of x; similarly $\phi_y(x, 0) = 0$. Hence $\phi_x(x, y)$ and $\phi_y(x, y)$ both have the property (i), and it is clear from (ii) that they also have the property (ii). Since $f(x, y)$ has these properties, it follows by induction that all the partial derivatives have them, and that these derivatives exist at every point.

It may be noted that this function satisfies conditions more stringent than those prescribed in the problem; namely the additional conditions that the mixed partial derivatives all exist, and are independent of the order in which the differentiations are performed.

Also solved by George Piranian, and the proposer.

Editorial Note. Piranian asks: What are the point sets E in the xy-plane for which there exists a function $(f(x, y; E)$ that is discontinuous everywhere on E and possesses all partial derivatives everywhere in the plane? For example, can E be everywhere dense? Can it be the entire plane?

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Analytic Geometry and an Introduction to the Calculus. By A. Clyde Schock and B. S. Warshaw. Prentice-Hall, Englewood Cliffs, N. J., 1960. 165 pp. $3.96.

This book is an attempt to present the material of the title in a form which the authors consider suitable for a high school class. The result is a weak selection of material supported by definitions which frequently are grossly incorrect and by incomplete demonstrations. A course based on this book can only result in a considerably weakened high school mathematics program and an impossible problem for the student who continues on to college. May this reviewer strongly recommend "What to do about a new kind of freshman," published in the August 1959 MONTHLY, on page 584, which details the problem which results.

DONALD A. NORTON
University of California, Davis

Introduction to Advanced Dynamics. By S. W. McCuskey. Addison-Wesley, Reading, Mass., 1959. viii+263 pp. $8.75.

The material covered in this book is split about equally between what is clearly elementary and what is usually regarded as advanced. The author states in the preface that the contents have been used as a one-semester course for seniors in physics, and feels that it paves the way for the study of modern physics or serves to introduce the nascent engineer or mathematician to domains of classical mechanics not treated in a first course in this subject.

Chapters 1, 3, 4, and part of 2 are elementary, and cover approximately the material on particle and rigid body mechanics contained in standard treatises on theoretical physics such as Page or Lindsay.

Chapters 2 (the latter portion), 5, 6, and 7 cover advanced material. In Chapter 2 we find Hamilton's principle, generalized coordinates, Lagrange's equations, non-holonomic and nonconservative systems, and a few pages on relativistic dynamics. To the reviewer this chapter seems misplaced, and should follow Chapter 4. Chapter 5 contains essentially the standard treatment of normal coordinates. A discussion of the stability of the linear system is found here, a rather novel feature in a book of this type. Chapters 6 and 7 get to the heart of the matter for the physicist, taking up Hamilton's equation, phase space, the Liouville theorem, canonical transformations, the Hamilton-Jacobi equation, and action and angle variables.

Criticisms are the following: The discussion of virtual work could be clarified. The notions of degrees of freedom and constraints of a system should be explained more fully before this principle is discussed. The definition of a position of stable equilibrium on page 46 is not clearly stated—at least for the mathematically inclined. On page 109, the author appears to use the terms moment and couple without distinction. From the discussion of the inertia tensor, it may be inferred (correctly) that this tensor depends upon the choice of origin of the coordinate system with respect to which the tensor is calculated. Since the parallel axis theorem is not mentioned, the important point cannot be made that it is really unnecessary to consider inertia tensors other than those associated with the center of mass. As the author admits in the preface, Chapter 5 is rendered somewhat cumbersome by the failure to use the matrix theory and notation.

The typography of the book is up to the usual high standard of its publishers; in particular, the figures are superb. Numerous exercises are given at the end of each chapter. Some of these are the well-worn academic standards, but a considerable number will be of real interest to the physicist and engineer.

This volume should serve very well as the basis for a terminal course in mechanics for the undergraduate engineering or mathematics student, and for a course prefatory to graduate work in modern physics.

WILLIAM H. PELL
National Bureau of Standards

How to Take a Chance. By Darrell Huff and Irving Geis. Norton, New York, 1959. 173 pp. $2.95.

This book is in the same light style as its predecessor *How to Lie with Statistics.* The illustrations are again by Irving Geiss and add a clever touch for the reader's enjoyment.

This new book presents a popularized approach to some applications of probability theory. In it the author dwells on points of a deeper nature than he did in his previous book. In fact, he touches on Type I errors, variances, confidence limits, sample size, validity of inferences, randomization and quite a number of others. They are, of course, not discussed in this terminology; indeed, a layman would never realize that he was being exposed to some of the basic concepts of statistical theory. Few of us could write in this style.

Specific applications of probability theory in this book are bridge, poker, molecular motion, game theory, E.S.P.—to mention just a few. The book also has quite a number of references to people in the various fields and consequently serves to acquaint the reader with some of the outstanding names.

Like all books on probability theory, it contains a section on problems and puzzles.

Do not expect a treatise on probability theory, but do read this book; you will enjoy it.

W. H. Williams
McMaster University

Abelian Groups. By L. Fuchs. Publishing House of the Hungarian Academy of Sciences, Budapest, 1958. 367 pp. $9.00.

The combination in one book of treatise and textbook is not a new phenomena in mathematical publishing. However, rarely has it been done with such felicity and success as in Fuch's book. Starting with the fundamental ideas, the book carries the beginning student through all the principal areas such as cyclic groups, divisible groups, pure subgroups, torsion and torsion-free groups, etc. At the same time—and particularly in the latter half of the book—the specialist will find an almost exhaustive treatment of current research in Abelian groups, *e.g.* the structure of additive groups of rings and multiplicative groups of fields.

The treatment throughout is detailed and extremely clear. The subject matter is so comprehensive that it might be advisable for teachers to point out to a beginning student those parts that might best be omitted on a first reading lest he be lost in too much detail. One of the happiest features of the book is the abundance of exercises and problems at the end of each chapter. The exercises are never trivial and range from the fairly direct to the quite complicated. The problems (86 in all) are all problems which were unsolved as of the writing of this book. This book should do a great deal to stimulate interest in these problems. A thorough bibliography is given. Finally, the book seems to be free from any major misprints.

A listing of the subject matter covered is as follows: Basic concepts; Direct sums of cyclic groups; Divisible groups; Direct summands and pure subgroups; Basic subgroups; The structure of p-groups; Torsion-free groups; Mixed groups; Homomorphism groups and Endomorphism rings; Group extensions; Tensor products; The additive group of rings; The multiplicative group of fields; The lattice of subgroups; Decompositions into direct sums of subsets; Miscellaneous questions.

E. H. BATHO
University of New Hampshire

Homotopy Theory. By S. T. Hu. Academic Press, New York. 1959, xiii+347 pp., $11.00.

With the exception of Hilton's *An Introduction to Homotopy Theory*, a part of Steenrod's *Topology of Fibre Bundles*, some seminar and course notes, and a few papers that contain expository material, there has been almost nothing available to provide an adequate and up-to-date background in one of the most vigorous fields of algebraic topology. Hilton's and Steenrod's books are somewhat condensed and have many gaps, whereas the notes and papers are usually quite specialized in their treatment of homotopy theory. Because of this, Professor Hu's book fills a definite need and is more than welcome.

Presupposing the spirit and much of the material of Eilenberg-Steenrod's *Foundations of Algebraic Topology*, the book gives a readable and modern introduction to homotopy theory. The first two chapters introduce the main problems of homotopy theory and illustrate their treatment. Fibre spaces and mapping spaces are discussed in Chapter III and the homotopy groups are defined geometrically and also axiomatized in Chapter IV. The elementary exact sequences and techniques of computing homotopy groups is discussed in Chapter V. Obstruction theory is considered in the next chapter and the cohomotopy groups in Chapter VII. The next two chapters are devoted to spectral sequences and their applications in homotopy theory and Chapter V discusses classes of abelian groups and the generalized Hurewicz theorem. The last chapter is devoted to the computation of some of the homotopy groups of spheres.

A noteworthy feature of the book is the numerous exercises. A few are exercises in the usual sense but most serve as a supplement to the text and a guide to the journal literature. Thus, for example, Steenrod squares, Eilenberg-MacLane complexes, Kan complexes, the generalizations of obstruction theory (developed in the text for finite cell complexes only) and other topics are outlined—with references—in the exercises.

Although some topics are omitted entirely, the book provides a good foundation for their study in the literature. On the whole, the book not only fills a need, it does so very well.

JOHN B. GIEVER
New Mexico State University

Statistical Estimates and Transformed Beta-Variables. By Gunnar Blom. Wiley, New York, 1959. 176 pp. $5.00.

This book presents certain results in the theory of transformed beta-variables, which are a generalization of the notion of order statistics. General properties of transformed beta-variables are discussed, including approximations to the first two moments and the asymptotic distribution of a linear combination. The construction of linear estimates of location and scale parameters in any continuous distribution is studied and a method is presented for finding "nearly best" linear estimates of these parameters.

The first two chapters present a theory of estimation generalized to cover asymptotic minimum variances of estimates for all continuous distributions. This general theory includes the Cramer-Rao inequality as a special case.

The next six chapters discuss moments and probability distributions for transformed beta-variables. Such variables include order statistics and the transformation of the binomial and related distributions to others which may be easier to handle, e.g. the normal distribution, as special cases. Means and variances are approximated for "nonsingular" and "singular" transformed beta-variables, and finally for linear combinations of these variables.

Chapters 9 through 14 discuss applications of the theory of transformed beta-variables to the theory of linear estimation. Here, the notion of "nearly-best" linear estimates is used. Comparisons are made with other methods of estimation. A discussion is given of the connection of these ideas with the problems of plotting points on probability papers.

Dr. Blom's book brings together known results in these fields, adds many results of his own and points out relationships among these ideas not generally realized. It is a useful addition to the reference literature of both order statistics and the theory of estimation.

PAUL D. MINTON
Southern Methodist University

The World of Ki. By John D. Goodell, Riverside Research Press, Greenwich, Conn., 1957. 215 pp. $1.00.

This is a curious and interesting combination of two books. One deals with the game of Go, originally Chinese (wei-chi) and later adopted by the Japanese. The other is a loosely connected sequence of essays on games and game-playing, treating such topics as thought, the origin of board games, games played by machines, special properties of games, game theory, the logical invention of games, complete information games, elements of luck, and possible applications of games to learning and teaching.

Go is a board game with very simple rules which can be learned in ten minutes, but is complex and subtle enough to present a considerable challenge to the highest intelligence. In this respect it compares very favorably with chess, and many devotees of both games would regard it as far superior. For the beginner,

this book has certain virtues, such as a clear exposition of the rules, given in several forms, and a discussion of many of the troublesome questions often raised in the early stages of learning. On the other hand, a disproportionate amount of attention and space is given to the relatively rare situations which may arise. The author might well have used some of this space to give the scores of several typical games together with annotations. While it is true that no one will learn to play well just by reading this—or any—book, the beginner would have been helped more by concrete examples than by wide generalizations. Also, assuming that the reader has become sufficiently interested in Go to wish to pursue it further, a reference to the American Go Association (96 Cedar Avenue, Hackensack, N. J.) would be appropriate. Equipment and books can be obtained through the Association. There is an excellent book for beginners by Takagawa (in English).

One may disagree with the author about some of his comments in the general sections, but they are always thought-provoking. Scattered throughout are descriptions of various esoteric games. One in particular caught this reviewer's fancy, the one described on pages 169 and 170 (called tawlbrdd in Wales, tablut in Lapland). It appears to be extremely interesting and worth trying. Anyone for tawlbrdd?

N. J. FINE
The Institute for Advanced Study

Lectures on Fourier Integrals. By Salomon Bochner. (Translated by Morris Tenenbaum and Harry Pollard.) Ann. of Math. Studies No. 42, Princeton University Press, Princeton, 1959. 333 pp. $5.00.

This edition of Bochner's well-known book is a faithful translation into English of the original (1932) German edition. A translation of the author's supplement on Monotone Functions, Stieltjes Integrals and Harmonic Analysis, originally published in German [Mathematische Annalen, vol. 108, 1933, pp. 378–410] appears as an appendix in this edition.

The material covered in the body of the book is indicated by the chapter headings: Basic Properties of Trigonometric Integrals, Representation and Sum Formulas, The Fourier Integral Theorem, Stieltjes Integrals, Operations with Functions of Class $\mathfrak{F}_0$, Generalized Trigonometric Integrals, Analytic and Harmonic Functions, Quadratic Integrability, and Functions of Several Variables.

The typing and its reproduction are both done well.

RUEL V. CHURCHILL
University of Michigan

Testing Statistical Hypotheses. By E. L. Lehmann. Wiley, New York, 1959. xiii+369 pp. $11.00.

Lehmann presents the testing of hypotheses in the framework of Neyman-Pearson theory. The closely related problem of confidence intervals is also in-

cluded. The author deals only briefly with the modern theory of statistical decisions and points out from time to time the resultant restrictions of his approach.

The book begins with a discussion of the general decision framework and basic probability theory. The next chapter deals with uniformly most powerful tests, the Neyman-Pearson fundamental lemma, and includes a section on sequential probability ratio tests. Next, come chapters on unbiasedness, invariance, linear hypotheses and finally, the minimax principle.

Measure theory is a requirement of the book and, strictly speaking, *no* knowledge of statistics is needed. However, in general, it would not be advisable to attempt to read this book without some background of statistical theory. Indeed, Lehmann recognizes that most readers will have such a background and as a result he has not included numerical illustrations.

This book would be excellent for an advanced text. It is clearly and concisely presented and has ample good problems for the student's consideration. The problems are divided into sections which correspond to the sections of the chapter.

The author has taken the very worthwhile effort to annotate most of the references cited. Consequently, this bibliography can be used as an excellent, accessible guide to further reading on the various topics.

This book is a significant contribution to statistical literature and will no doubt be one of the standard works in the area.

W. H. WILLIAMS
McMaster University

BRIEF MENTION

A Source Book in Mathematics. (Two volumes.) By David Eugene Smith. Dover, New York, 1959. xiii+701 pp. $1.85 each or $3.50 set.

This reprint of Smith's well-known 1929 book needs no review other than an indication of its availability.

A Course in Mathematical Analysis. (Three volumes.) By Edouard Goursat, translated by E. R. Hedrick. Dover, New York, 1960. 548+300+259 pp. $5.00 set.

The reprinting of the Hedrick and Dunkel translations of Goursat's work from the early 1900's should make these volumes easily available to many. Included in the series are: Volume I, Applications to Geometry, Expansion in Series, Definite Integrals, Derivatives and Differentials. Volume II, Part I, Functions of a Complex Variable. Volume II, Part II, Differential Equations.

Functions of a Complex Variable. The Theory of Functions of Real Variables, Volumes I and II. By James Pierpont. Dover, New York, 1960. 583+560+645 pp. $2.45 each.

Three additional early twentieth-century books which have been out of print for some time.

Mathematics: A House Built on Sand? By Ivan Niven. University of Oregon Books, Eugene, Oregon, 1959. 38 pp. $1.00.

This transcription of a lecture delivered at the University of Oregon could well be in every library, public and private. It provides an excellent popular (*i.e.*, nonmathematical) discussion of a series of important events in mathematical thought.

Ramanujan, 12 Lectures on Subjects Suggested By His Life and Work. By G. H. Hardy. Chelsea, New York, 1959. 236 pp. $3.95.

This Chelsea reprint of Hardy's 1940 work on the famous Indian number theorist, Ramanujan, will be welcomed by number theorists everywhere.

Business Mathematics, Exercises, Problems and Tests. By R. Robert Rosenberg. Gregg Publishing Division, McGraw-Hill, New York, 1959. 218 pp. $2.20.

It seems inconceivable that, when the *Harvard Business Review*, *Fortune*, and *Higher Education for Business* are advocating that business students be given some real mathematics, even calculus and beyond, a book such as this could appear. It is strictly a workbook in simple arithmetic. The last few pages contain a two-part final examination, the first part of which is mostly arithmetical. Section 2 of the final examination contains similar arithmetic, but is well loaded with business terms. Typical Problem: "A man sold for $9.00 a cigarette lighter that cost him $7.50. The markup was what part of the retail." (sic)

Hydrodynamics. By D. H. Wilson. St. Martin's Press, New York, 1959. viii+149 pp. $5.50.

A concise introduction to "classical" hydrodynamics based upon vector theory, which is assumed.

Probability and Statistics. The Harald Cramér Volume. Edited by Ulf Grenander. Wiley, New York, 1959. 434 pp. $12.50.

Those who prefer a more mathematical version of statistics may value this book of studies presented to Harald Cramér in honor of his 65th birthday. The authors of its 19 articles include some of the best-known mathematicians and mathematical statisticians in the country. Most of these articles will undoubtedly be reviewed in *Mathematical Reviews*, hence no extensive review is anticipated here.

Introduction to Theoretical Meteorology. By Seymour L. Hess. Holt, Rinehart and Winston, New York, 1959. xiv+362 pp. $8.50.

An excellent text in modern dynamic meteorology. The preface claims that no mathematical demands other than a thorough knowledge of differential and integral calculus are made; however, the author is apparently well-grounded in mathematics and uses partial differential equations freely. How many of our readers can derive formulas to determine how fast a billiard ball 2 inches in diameter must be propelled directly at another billiard ball 102 inches away in order that it shall just barely miss striking the second ball, due to Coriolis force, if the table is located at 43° North latitude? (See problem 6, page 174).

Introduction to Mathematical Thinking. By Friedrich Waismann. Harper, New York, 1959. $x+260$ pp. $1.40.

This is a new printing of Theodore J. Benac's translation of Waismann's 1951 work. The purpose of the book is "to give an insight into the nature of mathematical concept formation, that is, to point out the activities of the mathematician that might be of interest to a philosophically minded observer."

1960 Catalog—Educational Motion Pictures. Audio-Visual Center. Indiana University, Bloomington, Indiana, 1960. 664 pp.

This appears to be a reasonably careful compilation of 16 mm educational motion pictures. By checking through various portions of the subject matter index as suggested under the general title "Mathematics" a fairly extensive list of mathematical films can be compiled; a useful reference in these days of urgent requests from high school teachers on how to spend Title III money of the Defense Education Act. This may be an appropriate time to again mention that supplementary library books in mathematics for the enrichment of high school curriculum are also approved for purchase under this act.

Qualifications and Teaching Loads of Mathematics and Science Teachers, in Maryland, New Jersey, and Virginia. By Kenneth E. Brown and Ellsworth S. Obourn. U. S. Department of Health, Education, and Welfare Circular No. 575. U. S. Government Printing Office, Washington, D. C., 1959. viii+101 pp. 70¢

A careful compilation of statistics from the three states named, to determine the background preparation, teaching load and teaching combinations of mathematics and science teachers. This study could well serve as a pattern for similar studies in other states.

Basics of Digital Computers. (Three volumes.) By John S. Murphy. John F. Rider, New York, 1958. 116+133+136 pp. (soft covers) $2.80 each; three volumes in a single cloth binding, $9.50.

An easy-to-read "picture book" which provides in three volumes an excellent introduction to some of the basic ideas of computer hardware, both past and present.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Butler University: Assistant Professor Juna L. Beal retired June 6, 1960; Mr. Robert Lamberson and Mrs. Joy F. Martin have been appointed Instructors.

College of the Holy Cross: The Reverend R. J. Swords has been appointed President of Holy Cross; Professor V. O. McBrien has been appointed Chairman of the Department of Mathematics; Professor J. A. Nestor will serve as Acting Chairman during the academic year 1960–61; Dr. D. L. McQuillan, Johns Hopkins University, has been appointed Assistant Professor; Assistant Professor D. G. Dewey, Aquinas College, has been appointed Assistant Professor.

Cornell University: Dr. Zbigniew Ciesielski, Assistant in Mathematics, University of Pozan, has been appointed Instructor; Dr. R. C. Curjel, Research Assistant, Swiss Federal Institute, has been appointed Instructor; Assistant Professor R. V. Chacon, University of Wisconsin, has been appointed Visiting Assistant Professor ; Dr. Leonard Gross, Yale University, has been appointed Assistant Professor; Assistant Professor S. V. Parter, University of Indiana, has been appointed Assistant Professor of Mathematics

and Associate Director of the Computing Center; Assistant Professor Jerome Sacks, Columbia University, has been appointed Acting Assistant Professor; Dr. S. J. Taylor, University of Birmingham, has been appointed Visiting Assistant Professor.

University of Minnesota, Department of Statistics: During the academic year 1958–1959 the Department of Statistics was established at the University of Minnesota. Following is the current staff of the Faculties of Statistics: Statistics: L. Hurwicz, I. Olkin, D. Richter, I. R. Savage, N. Sobel; Mathematics: G. Baxter, M. Donsker, B. Lindgren, S. Orey, W. Pruitt, E. Reich, F. Spitzer; Agriculture: R. Comstock, C. Gates; Biostatistics: J. Bearman, J. Berkson, B. Brown, E. Johnson, R. McHugh; Business Administration: D. Hastings, J. Neter; Industrial Engineering: G. McElrath.

University of Puerto Rico: Professor Mariano Garcia has been appointed Dean of Studies, Rio Piedras Campus; Professor Juan de Dios Quinones has been appointed Chairman of the Department of Mathematics, Mayaguez Campus.

Worchester Polytechnic Institute: Professor R. N. Cobb has been named John E. Sinclair Professor of Mathematics; Professor E. L. Buell has been appointed Head of the Department of Mathematics; Dr. W. J. Hardell, Radio Corporation of America, Moorestown, New Jersey, has been appointed Assistant Professor; Mr. B. C. McQuarrie, Wasatch Academy, Mount Pleasant, Utah, has been appointed Instructor.

Associate Professor A. G. Azpetitia, University of Massachusetts, has been appointed Visiting Professor at Brown University.

Brother Joseph Heisler, Archbishop Curley High School, Miami, Florida, has been appointed Instructor at St. Edward's University.

Dr. S. D. Chatterji, Michigan State University, has been appointed Lecturer, University of South Wales, Kensington, New South Wales, Australia.

Assistant Professor N. H. Chokey, Johns Hopkins University, has been appointed a member of the Senior Staff, Applied Physics Laboratory, Johns Hopkins University.

Dr. J. D. Daugherty, Fairleigh-Dickinson University, has been appointed Professor at State Teachers College, Kutztown, Pennsylvania.

Associate Professor Harley Flanders, University of California, Berkeley, has been appointed Professor at Purdue University.

Dr. J. H. Hodges, Cornell Aeronautical Laboratory, Buffalo, New York, has been appointed Assistant Professor at the University of Colorado, Boulder, Colorado.

Dr. Constantine Kassimatis, Cornell University, has been appointed Associate Professor at North Carolina State College.

Mr. E. H. Kingsley, Chief, Operations Analysis Office, Headquarters, OCAMA, Tinker AFB, Oklahoma, has accepted a position as Analyst in the Operational Science Laboratory of the Research Triangle Institute, Durham, North Carolina.

Dr. J. H. Lindsay, University of British Columbia, has been appointed Lecturer at the University of Toronto.

Assistant Professor Gabriel Margulies, Florida State University, has been appointed Resident Director of the Florida State University Graduate Center, Elgin Air Force Base, Florida.

Associate Professor R. A. Struble, North Carolina State College, has been promoted to Professor.

Dr. J. S. White, Aeronautical Division, Minneapolis-Honeywell Regulator Company, has accepted a position as Senior Research Mathematician with the Research Laboratories of the General Motors Technical Center, Warren, Michigan.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of the General Mathematics Examination of the

1960 Preliminary Actuarial Examinations are as follows:

FIRST PRIZE OF $200

Gitlin, Todd A.	Harvard University

ADDITIONAL PRIZES OF $100 EACH

Emerson, William R.	California Institute of Technology
Goodman, Richard H.	Harvard University
Landman, Maurice A.	Harvard University
Lorden, Gary A.	California Institute of Technology
McDonnell, Robert N.	University of Chicago
Newmeyer, John A.	California Institute of Technology
Sampson, Schuyler S.	Bowdoin College
Shulsky, Abram N.	Cornell University

The Society of Actuaries has authorized a similar set of nine prizes for 1961.

Beginning in 1961, the Preliminary Actuarial Examinations will consist of two examinations: The General Mathematics Examination (based on the first two years of college mathematics), and The Probability and Statistics Examination.

The 1961 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians, and will be administered by the Society of Actuaries at centers throughout the United States and Canada on November 16, 1960 and May 10, 1961. The closing date for applications for the May examination is April 1, 1961.

Further information concerning these Examinations can be obtained from the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 112 persons have been elected to membership by the Board of Governors on applications duly certified.

GERALD E. ABRAHMS, B.A. (Colorado) Seismic Computer, The Atlantic Refining Company

ALBERT S. ANDERSON III, Student, Emory University

DALE E. ANDERSON, Student, University of Nebraska

WILLIAM N. ANDERSON, JR., Student, Carnegie Institute of Technology

JACK L. BALDWIN, Student, Southwest Missouri State College

MARILYN MAE BALDWIN, Student, duPont Manual High School, Louisville, Kentucky

MICHEL L. BALINSKI, Ph.D. (Princeton) Instr., Princeton University; Consultant, Mathematica

ROBERT P. BALLES, Student, Fordham University

JAMES M. BARKO, Student, St. Vincent College

HAROLD M. BELL, B.S. (Kentucky) Grad. Student, Eastern Kentucky State College

RUDOLPH J. W. BERAN, Student, St. Michael's College School

JOSEPH B. BOHAC, B.S. (St. Louis) Engr., Horner and Shifrin Consulting Engineers

DON LOREN BOHMONT, B.S. (Nebraska) Teaching Asst., Colorado State University

HOWARD I. BOWN, B.S. (Beloit Coll.) Grad. Student, Beloit College

DENNISON R. BROWN, B. S. (Duke) Asso. Math., Louisiana State University

EUGENE B. BRUNNER, Student, Indiana State College

ROBERT S. BURDICK (M.I.T.) Teacher, Iolani School, Honolulu, Hawaii

EDWARD A. BURFINE, Student, California State Polytechnic College

PHILIP R. CARLSON, B.S. (Minnesota) Grad. Student, University of Minnesota

RUU-KWANG CHANG, M.S. (Colorado) Teaching Asst., Colorado State University

WARREN E. CHASE, B.S. (Franklin and Marshall Coll.) Grad. Student, Franklin and Marshall College

FRANKLIN D. CHEEK, Student, University of Wisconsin

RUDOLPH J. CHERKAUER, Ed.D. (Buffalo) Prof., State University of New York, College of Education, at Buffalo

HERMAN CHERNOFF, Ph.D. (Brown) Prof., Stanford University

ROBERT E. CLAPP, Student, Reed College

MRS. EMAGENE W. CLOTHIER, B.S. (Marquette) Teacher, Divine Savior High School, Milwaukee, Wisconsin

LEONARD COHEN, Ph.D. (Columbia) Instr., College of the City of New York

GORDON L. COLE, M.A. (Baltimore) Asst. Prof., Indiana Technical College

CRAIG COMSTOCK, B.E.A. (Cornell) Instr., U. S. Naval Postgraduate School, Monterey, California
JOHN S. CONNER, B.S. (U.S. Military Acad.) Grad. Student, Monterey Peninsula College
DONALD W. COUNSELLOR, Student, Rutgers College of South Jersey
WILLIAM C. CUTLER, B.S. (Sanford) Menlo Park, California
CARL P. DEAN, B.S. (Oklahoma) Instr., University of Oklahoma
ROBERT E. DODGE, JR., M.A. (Missouri) Instr., University of Missouri
JOANNA M. ENZMANN, B.S. (M.I.T.) Engr., Sylvania Electric Products
VOIGT M. EPTING, M.S. (S. Carolina) Superintendent, Pacolet Schools, South Carolina
LLOYD R. ERICKSON, JR., Student, Rice University
BRUCE FELDMEYER, Student, Sir Francis Drake High School, San Anselmo, California
MARIE FEORANZ, M.A. (N.Y.U.) Teacher, Lodi High School, New Jersey
JOSEPH C. FERRAR, Student, Michigan State University
SHERIDAN K. FERRIERA, A.A. (Sacramento Jr. Coll.) Math. Analyst, Aerojet-General Corp.
OSBURN R. FLENER, M.S. (Illinois) Teacher, Western Kentucky State College
HAROLD W. FLESNER, B.S. (S. Dakota) Grad. Student, South Dakota State College
MRS. MARIAN E. FLESNER, M.S. (S. Dakota) Instr., South Dakota State College
THOMAS W. FORGET, B.S. (St. Johns) Research Engr., Lockheed
JOHN H. FOX, JR., B.A. (U.C.L.A.) Math. Consultant, The Martin Company
JOSEPH A. FRISBIE, A.S. (Danbury) Lab. Tech., Barden Corporation
SEYMOUR GOLDBERG, Ph.D. (California) Asst. Prof., New Mexico State University
RICHARD L. GOODRICH, A.B. (Harpur) Programmer, I.B.M., Owego
BERNICE M. GORDON, M.A. (Minnesota) Classroom Consultant, Edina-Morningside Public Schools, Minnesota
ROBERT B. GRAFTON, Sc.B. (Brown) Lt. J.G., U. S. Navy
PRANAS A. GRUSAS, M.A. (Syracuse) Staff Mathematician, I.B.M., Bethesda
JAMES E. HARDIN, M.Ed. (Trinity) Teacher, MacArthur High School, San Antonio, Texas
JAMES F. HAWLEY, B.A. (Kansas) Grad. Asst., Kansas State College
ROBERT W. HEATH, Ph.D. (N. Carolina) Asst. Prof., University of Georgia
JOYCE HOCHSTEIN, Student, Hunter College
RONALD L. HOFFMANN, Student, duPont Manual High School, Louisville, Kentucky
FREDRICK A. HOPPERSTEAD, Student, North Texas State College
MRS. LUCILLE A. HORTON, M.A. (N.Y.S. Teachers Coll.) Head of Dept., Tri-Valley Central School, Grahamsville, New York
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DANIEL R. JESSUP, Student, Kent State University
RICHARD R. KARCH, Student, Case Institute of Technology
TERENCE J. KELLY, Student, University of Maryland
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DAVID H. KNOWLES, M.A. (San Jose) Dept. Chairman, Samuel Ayer High School, Milpitas, California
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FRED KRAKOWSKI, D. Sc. (Swiss Fed.) Asst. Prof., University of California, Davis
RONALD K. LACEY, Student, California State Polytechnical College
JOHN D. LEADLEY, M.S. (Washington) Asst. Prof., Reed College
PAUL AARON LEE, M.B. (Mukden) Medical Resident, St. Joseph's Hospital, Bronx
CONSTANCE J. LEUER, Student, Beloit College
PAUL J. LOATMAN, B.S. (Siena) Chief, Computer Unit, Watervliet Arsenal
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WARREN B. MCCLANAHAN, Student, Alabama College
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HAROLD D. PENDERGRAFT, Student, University of Oklahoma
CARL A. PERSINGER, Student, Virginia Polytechnical Institute
MRS. CHARLOTTE E. PICKETT, B.S. (Auburn) Asso. Engr., Boeing Airplane Company
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WILLIAM H. ROW, JR., Student, Knox College
N. JAMES SCHOONMAKER, Ph.D. (Pittsburgh) Prof. and Chairman of Dept., University of Vermont
SARAH SCHWARTZ, B.A. (Hunter) Teacher, Rabbi Jacob Joseph School, New York City, New York
RICHARD M. SHANK, A.B. (U.S.C.) Analyst, Hughes Aircraft
JAMES R. SILVA, Ed. M. (Harvard Head of Dept., Pelham High School, New York
ANNE T. SILVERSTEIN, M.A. (Columbia) Brooklyn, New York
WILLIAM S. SIMS, B.Sc. (Wales) Instr., Ryerson Institute of Technology
SISTER MARY JOHN M.S. (Marquette) Teacher, Father Ryan High School, Nashville, Tennessee
GASTON SMITH, M.A. (Alabama) Asst. Prof., Mississippi Southern College
CHARLES H. TAFT, M.S. (Michigan) Branch Manager, I.B.M., Corpus Christi
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OLEN J. WARD, JR., Student, Jacksonville, University
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LONNIE WHITEHEAD, JR., B.A. (Emory) Statistician, Emory University
FREDERICK W. WILKE, M.A. (Washington) Instr., University of Missouri
LAWRENCE A. WOLD, B.A. (Connecticut) Teaching Asst., University of Washington
WINSTON L. WONG, B.A. (San Jose State) Grad. Student, San Jose State College

THE NOVEMBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held on November 14, 1959 at Concordia College in Moorehead, Minnesota. Professor Jerry Heuer of Concordia College, Section Chairman, presided at the

morning session. Professor Sigurd Mundhjeld of Concordia College presided at the afternoon session. There were 50 people registered for the meeting, including 37 members of the Association.

During the business meeting the secretary read a report from the Minnesota National Laboratory for the Improvement of Secondary School Mathematics. The report discussed the past and future plans of the Laboratory which is under the direction of Professor P. C. Rosenbloom.

The following papers were presented:

1. *Estimation of correlation coefficients from scattergrams*, by Professor C. R. Perisho, Mankato State College.

If a scattergram is plotted from a bivariate distribution, the area covered by the points is, in general, elliptical. The correlation coefficient can be estimated from the shape of any of the elliptical contour lines. A formula is developed which gives the correlation coefficient if the angle of inclination of the major axis and the ratio of the major and minor axes of the ellipse are known.

2. *The generalized tower of Hanoi problem*, by Mr. Don Olivier, Carleton College, introduced by Professor F. L. Wolf, Carleton College.

The classical problem (a number of disks of different sizes are stacked on one of three needles; they must be transferred to another by a series of moves where only one disk may be moved at a time and a larger disk may never rest on a smaller disk) is generalized to use m disks and n needles. On the basis of a preliminary simplifying assumption (which is plausible but has yet to be proved) a formula for the minimum number of moves required to effect the transfer is derived and proved inductively.

3. *Euler's conjecture*, by Dr. E. T. Parker, Remington-Rand Univac (by invitation of the Executive Committee).

Euler conjectured in 1782 that there exists no pair of orthogonal Latin squares of any even order not a multiple of four. Earlier this year Professors R. C. Bose and S. S. Shirkhande and the speaker constructed pairs of squares of all orders $4t+2 \geqq 10$. (Papers are in Proc. Nat. Acad. Sci., May and June, 1959. Longer papers are pending.) Some history of the problem is presented, and the recent work is summarized. Particular reference is made to the connection with finite projective planes.

4. *Other results on orthogonal Latin squares*, by Professors N. S. Mendelsohn, S. Nathan, A. L. Dulmage, and D. M. Johnson, University of Manitoba.

Using as a basic square a direct product of cyclic squares, a method of constructing pairwise orthogonal Latin squares by means of permutations of rows is obtained. The results lead to larger numbers of squares than that predicted by the now defunct MacNeish conjecture. The squares are quite different from those constructed by Parker, Shrikhande and Bose.

F. L. WOLF, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky on April 30, 1960. Professor W. C. Royster, Chairman of the Section, presided at both the morning and afternoon sessions. Seventy persons attended the meeting, including fifty-seven members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor A. G. Anderson, Western Kentucky State College; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor A. W. Goodman, University of Kentucky.

By invitation, Professor W. J. Thron of the University of Colorado delivered an hour address "*Convergence Criteria for Continued Fractions*" at the afternoon session.

The following papers were presented:

1. *On the applications of B_0-spaces to summability theory*, by Mr. H. R. Coomes, University of Kentucky.

A discussion of the work of Zeller together with some new criteria in order that $B(AX) = BA(X)$ for matrices A and B and sequences X.

2. *A note on the convergence of* $1+\sum_{n=0}^{\infty}\binom{m}{n}x^n$ *for* $x=\pm 1$, by Professor Aughtum Howard, Eastern Kentucky State College.

The author gives an elementary proof of the convergence of the binomial series $1+\sum_{n=0}^{\infty}(-1)^n\binom{m}{n}$, $0<m<1$, and of $1+\sum_{n=0}^{\infty}\binom{m}{n}$, $-1<m<0$.

3. *On the fundamental theorem of linear equivalences in kinematic relativity*, by Professor W. S. Krogdahl, University of Kentucky.

4. *On half-groups*, by Professor Frank Levin, University of Kentucky.

5. *Concerning the Euler differential equation*, by Miss Helen S. Weihe, University of Kentucky.

This talk was concerned with solving the Euler equation by means of the substitution $y=(ax+b)^m$.

6. *On the Visiting Lecturer Program*, by Professor T. J. Pignani, University of Kentucky.

7. *Revisions in the mathematics curriculum at Western Kentucky State College*, by Professor A. G. Anderson, Western Kentucky State College.

Changes in the mathematics offerings at Western Kentucky State College, to become effective next fall, were outlined. Emphasis is on providing separate programs at the first year level for students majoring and minoring in mathematics and/or science and the group categorized under the general heading "non-science" students. At the junior-senior level, a similar separation is made to provide distinct programs meeting the needs of the group preparing for positions in secondary education and those having as objectives positions in business or industry or further training in mathematics at the graduate level.

8. *Generalization of an ordered group*, by Mr. W. K. Cope, University of Kentucky.

V. F. COWLING, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-seventh annual spring meeting of the Indiana Section of the Mathematical Association of America was held Saturday, May 7, 1960 at Earlham College, Richmond, Indiana. Approximately 50 members of the Association attended. Professor Kermit Carlson of Valparaiso University, Chairman of the Section, presided at both the morning and afternoon sessions.

The following officers were elected: Chairman, Professor M. E. Shanks, Purdue University; Vice-Chairman, Professor J. E. Yarnelle, Hanover College; Secretary-Treasurer, Professor P. T. Mielke, Wabash College.

Professor J. C. Polley reported on the activities of the State School and College Committee for Mathematics. This is a joint Committee of the Indiana Section of the Association, the Indiana Academy of Science, and the Indiana Council of Teachers of Mathematics. The purpose of the committee is to foster cooperation between school and college mathematics teachers and the State Department of Public Instruction. It was announced that a grant had been received from the National Science Foundation to support committee activities for the coming year.

A report on the National Test sponsored by the Association and the Actuarial Society, prepared by Professor Jack Forbes, was presented. It was decided that next year the Indiana School and College Committee would sponsor this test within the state.

Professor R. H. Bing of the University of Wisconsin presented the guest lecture on *Recent Developments in Euclidean Topology*.

The following short papers were presented:

1. *The Earlham statistics program*, by Professors Roland Smith and Howard Alexander, Earlham College.

2. *Trends in teacher education*, by Professor Charles Brumfiel, Ball State Teacher's College.

3. *The additive group of real numbers*, by Professor Johannes DeGroot, Purdue University.

4. *Approximate solutions of the Dirichlet problem*, by Professor Donald Greenspan, Purdue University.

5. *On repeated integrals*, by Professor J. H. B. Kemperman, Purdue University.

6. *The recursive relation in Pascal's triangle*, by Professor Dale Mesner, Purdue University.

CHARLES BRUMFIEL, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the Mathematical Association of America was held on May 7, 1960 at St. Olaf College in Northfield, Minnesota. Professor C. S. Carlson of St. Olaf College presided at the morning session. The Section Chairman, Professor Jerry Heuer of Concordia College, presided at the afternoon session. There were 68 registered for the meeting, of whom 55 were members of the Association.

At the business meeting Professor Seymour Schuster of Carleton College reported on the 1960 Minnesota High School Mathematics Contest which was sponsored, in part, by the Minnesota Section. 261 high schools and 9,655 students participated in the contest this year. The thanks of the section were expressed to Professor Schuster and his committee for doing an excellent job in handling the contest.

It was suggested that it would be helpful to the colleges and high schools in Minnesota if the Association discovered and published the natures of the available mathematics programs at the colleges and universities in the state. It was moved and passed that a committee be appointed to study this matter.

The motion was made and passed to accept with thanks a donation made to the section for the purpose of purchasing name tags to be used at section meetings. The donation was made by Dr. E. J. Parker of Remington-Rand Univac.

The following officers were elected to serve during the school year of 1960–61: Chairman, Professor Fulton Koehler of the University of Minnesota; Secretary-Treasurer, Professor Murray Braden of Macalaster College; Members of the Executive Committee, Professor O. E. Stanaitis of St. Olaf College and Professor Warren Thomsen of Mankato State College.

The following papers were presented:

1. *Density theorems for sets of integers, and a generalization*, by Professor William Moser, University of Manitoba. (by invitation of the Executive Committee).

Let $f(x)$ be defined, integrable and $0 \leqq f(x) \leqq 1$ on $[0, 1]$; $g(x) \equiv 1 - f(x)$ on $[0, 1]$; $f(x) \equiv g(x) \equiv 0$ outside $[0, 1]$. Furthermore let $\int_0^1 f(x)dx \geqq \alpha t$, $0 \leqq t \leqq 1$, α a fixed real positive number, $0 < \alpha < 1$. Let $D(t) = \int_0^1 f(x)g(x+t)dx$ and $D = \inf_{f(x)} \sup_t D(t)$. Bounds for D are found. The methods used and the results obtained are generalizations of similar results regarding the density of the sum of two sequences of integers (See Kasch, Math. 3 vol. 62, 1955 pp. 368–387). Related problems are discussed and a brief outline of the history of these problems is given.

2. *An obvious—but untrue—algebraic result related to geometry*, by Professor Seymour Schuster, Miss Alison Krotter, and Mr. Don Olivier, Carleton College.

A paper, *The lines and planes connecting the points of a finite set*, by T. Motzkin (Trans. Amer. Math. Soc., 1951), contained the following statement: "Let $\{a\}=\{b\}=\{c\}$ be finite subsets of field F without 0 such that every product ab is in $\{c\}$, every bc in $\{a\}$, every ca in $\{b\}$; a simple computation shows that one of them, say $\{a\}$, consists of the m mth roots of 1 for some m, and either $\{b\}=\{c\}=\{a\}$ or $\{b\}=\{c\}=-\{a\}$." The authors, believing that no really "simple computation" exists, exhibit two proofs of the first part of Motzkin's assertion, and show that the second part is true only if m is odd. If m is even, then either $\{b\}=\{c\}=\{a\}$ or $\{b\}=\{c\}=ae^{\pi i/n}$)

3. *On the definition of surface area*, by Professor James Serrin, University of Minnesota.

It is a common opinion that the well-known formula for surface area cannot be proved rigorously except by extremely delicate arguments. In consequence, calculus textbooks tend to give only heuristic justifications of this formula. Beginning with Lebesgue's definition of area by means of approximating polyhedral surfaces, however, it is actually not difficult to prove that the area of a piecewise continuously differentiable surface $z=f(x, y)$ is given by the usual formula. Indeed, this fact follows at once from a simple inequality which (roughly speaking) asserts that averaging is an area-shrinking operation.

F. L. WOLF, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-fourth annual meeting of the Ohio Section of the Mathematical Association of America was held at Kent State University, Kent, Ohio, on Saturday, May 7, 1960. Professor W. R. Van Voorhis, Chairman of the Section, presided at the morning and afternoon sessions. There were 84 persons registered in attendance, including 71 members of the Association.

Officers elected for the coming year are: Chairman, Professor Wade Ellis, Oberlin College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor Wendell Johnson, Hiram College; Program Committee, Professor C. W. Topp, Fenn College, Chairman, Professor W. E. Restemeyer, University of Cincinnati, and Professor Clarence Heinke, Capital University.

The following papers were presented:

1. *The impact of modern mathematics upon college teachers*, by Professor W. R. Van Voorhis, Fenn College (Chairman's Address).

The growing emphasis upon mathematics in our secondary and college programs calls for critical evaluation of course content, and changes in instructional methods and requirements. Soon course offerings may depart from the conventional listings of specific topics and be presented under broad headings such as mathematical structures, models, processes, transformations, etc. Instructional methods for adequately handling larger numbers of students with more extensive and variable backgrounds will be found by the teachers of mathematics rather than by those trained primarily in methodology. There will be greater emphasis upon independent study under the supervision of experienced instructors with the aid of assistants.

2. *A procedure for optimizing a function involving uncertain parameters*, by S. K. Gupta, Case Institute of Technology.

Exact values of parameters appearing in mathematical decision models are seldom, if ever, known and must be estimated whenever a given decision function is to be used. Traditional statistical techniques of estimation generally fail to take into account consequences of the effect of overestimation and underestimation of the parameters. Consequently, it is desirable to establish procedures for using estimates in such a way that the "cost of error" resulting from their use is reduced. Pragmatically, then, it is important to find a procedure which will minimize the expected cost of error and/or its variance. The nature of a general solution to this problem is discussed, and a specific method of *adjusting* parameter estimates is developed. In particular, techniques are given for models involving one parameter, and suggestions are given for extending these techniques to the cases of two or more parameters.

3. *Reflections in plane geometry*, by Professor P. B. Yale, Oberlin College.

Elementary congruence and incidence relations between points and lines in the Euclidean plane are characterized by group theoretic relations between the corresponding point and line reflections. Proofs of some theorems of elementary geometry are given using these characterizations.

4. *Remarks on some problems in the American Mathematical Monthly*, by Professor C. F. Pinzka, University of Cincinnati.

Several unsolved problems of an elementary nature were discussed. The birthdays problems, requiring the minimum number of people n such that the probability of a birthday coincidence exceeds $\frac{1}{2}$, is known to have the solution $n=23$. If the problem is extended to the cases of sets of 3, 4, 5 people with the same birthday, then a Poissonian approximation yields the values $n=88$, 187, 314, respectively, the first of which is easily verified by the formula

$$p(n) = 1 - (365!n!/365^n) \sum_{k=0}^{[\frac{1}{2}n]} 1/(2^k k!(n-2k)!(365-n+k)!).$$

5. *Quantum theory of gravitation*, by Professor S. N. Gupta, Wayne State University.

The conflict between Einstein's theory of gravitation and quantum mechanics can be resolved by following a new approach, which is based on the fundamental assumption that the physical space is flat and not curved. It can be shown that Einstein's gravitational field questions can be transformed from curved space to flat space by means of a Lorentz-covariant expansion. It is then possible to quantize the gravitational field, and the resulting theory predicts the existence of a new type of elementary particle called the graviton. The quantum theory of gravitation also throws new light on several physical phenomena.

6. *Mathematics—Is it a prerequisite to a college degree?* by Professor E. A. Tabler, The University of Akron.

Many surveys disclose a lack of precollege accomplishment in the quantitative language of mathematics. One such national survey concludes, "In our elementary and secondary schools, the coverage in mathematics is inadequate and the too little that is required is not well taught." Believing that college graduates need proficiency in mathematical concepts to be encountered, that mathematics is not just a prerequisite to certain other subjects, but a language, prerequisite for being efficient citizens in a modern world, The University of Akron includes Numbers Communication in its long-established General Education Program. Evaluation procedures indicate a needed service is being provided.

7. *Curvature without the second derivative*, by Professor L. C. Graue, Bowling Green State University.

Methods for finding curvatures of plane curves without involving the second derivative are presented. These are applied to extending the notion of curvature to some examples where the second derivative does not exist.

8. *Twelfth-grade mathematics program at Hiram College*, by Professor M. A. Moore, Hiram College.

Hiram College has for two years sponsored a twelfth-grade mathematics program for gifted students of nearby small high schools. The program was supported by the Wean Foundation and by N. S. F. The course blended algebra, analytic geometry, and trigonometry, with about equal emphases on the three subjects. The class met daily on the Hiram campus, the students furnishing their own transportation. Problems that arose involved road conditions, variation among participating schools in holidays and field trips, etc. Both classes maintained a good level of interest and attainment; those who continued into college unanimously report that the course was valuable precollege training.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The sixteenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on May 7, 1960. Professor Dis Maly, the Chairman of the Section, presided at the morning session and Professor B. H. Gere, Vice-Chairman of the Section, presided at the afternoon session. There were 122 persons in attendance,including 81 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor B. H. Gere, Hamilton College; Vice-Chairman, Professor D. E. Kibbey, Syracuse University; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. The Committee on the Strengthening of Mathematics in the Section was discharged with thanks. Professor Nura Turner, of the State University College of Education at Albany, as Chairman of the Contest Committee, reported another successful contest. She also reported the plans for bringing to Albany on May 14, 1960 for a day of lectures and discussions on Mathematics as a Career the students in the upstate section of New York who placed in the top one percent in the 1960 contest.

The program was as follows:

1. *A problem in probability*, by Professor A. G. Davis, Clarkson College of Technology.

The speaker evaluated the probability that a triangle can be formed with length of sides equal to numbers x_i obtained by drawing from a uniform distribution 0 to 1.

2. *Approximately continuous functions*, by Professor A. G. Fadell, University of Buffalo.

A Lebesgue-measurable subset S of Euclidean n-space E_n is called a *density set* if every point of S is a point of density of S. Then the class of density sets D_n topologizes E_n. Hence, approximately continuous functions become continuous relative to the density set topology. Theorems on continuous functions for general topological spaces therefore apply to approximately continuous functions. A few such theorems are cited.

3. *Lagrange's theorem and stationary functions*, by Mr. R. J. Pegis, Bausch and Lomb Optical Company.

Methods are given for evaluating a function when it is stationary with respect to one or more of its variables. It is shown through an extension of Lagrange's Theorem that we do not need to know the values of the variables which make the function stationary. Applications are indicated, especially in optics.

4. *Methods to treat mathematical problems with symmetry of rotation*, by Dr. M. J. Herzberger, Eastman Kodak Company.

Given two vectors $\vec{b}_1$ and $\vec{b}_2$ and their symmetric functions e_1, e_2, e_3; given a function E of the e_i developed into a power series of the e_i; a method is described for computing the coefficients of E for the symmetric functions of B_1 and B_2 where $\vec{B}_1=\alpha\vec{b}_1+\beta\vec{b}_2$, $\vec{B}_2=\gamma\vec{b}_1+\delta\vec{b}_2$, and α, β, γ, δ are general functions of the e_i. The method utilizes certain polynomials first developed by T. Smith. The solution has many applications in the field of geoemtrical optics as well as in all problems connected with the extremals of a variation.

5. *A useful course in geometry for future high school teachers*, by Professor Israel Halperin, Queen's University.

The current agitation to modernize the high school mathematical curriculum by introducing modern concepts such as those of set theory, may, in practice, lose more than it gains. The advantages of the old-fashioned material should be well understood; namely, good students, at least, are brought face to face with difficult problems which they have the ability and the mathematical equipment to solve. Foremost in geometry should be the challenging problem, correctly posed. The speaker's course for future teachers, based on these ideas, follows closely the axiomatic development given in Hilbert's "Grundlagen der Geometrie."

6. *Mathematics in industrial research*, by Dr. E. W. Marchand, Eastman Kodak Company.

The speaker emphasized the wide gulf which separates the fields of pure and applied mathematics, the great diversity of talents an industrial mathematician should have, and the inadequacy of a classical mathematical education in preparation for industrial mathematics.

7. *The case for a general education course in mathematics*, by Professor Harriet F. Montague, University of Buffalo.

This paper describes the results of a study made to determine relations between grades received in a course entitled Perspectives in Mathematics taught to some 500 students from 1951 to 1959 and such items as previous training in mathematics, rank in high school graduating class, and beginning of course attitudes toward mathematics, such as fear, indifference, enjoyment.

8. *Teaching experiences with a high school supplementary mathematics program*, by Mr. G. R. Rising, Greece Olympia High School.

A program of mathematics enrichment for selected students from four Monroe County secondary schools is described with special reference to problems in administration of the program, selection of topics to be taught and evaluation of the results.

9. *A modification of Birkhoff's axioms for Euclidean geometry*, by Professor Roland Brossard, Universite de Montreal.

As a result of a weakening of the axiom of angular measure an angle can be defined without any orientation. However, it now has two measures, α and $2\pi-\alpha$, the smallest called the interior measure and the other, the exterior measure. The fact that the measures of a straight angle are both π is taken as an axiom. To permit the construction of space geometry the notion of plane bundle of half-lines is introduced, and the axioms on angles are formulated in terms of plane bundles of half-lines.

10. *Some elemtary aspects of integral geometry*, by Professor W. R. Baum, Syracuse University.

The concept of figure-measure, in particular line-measure, is discussed from geometric and analytic points of view and illustrated by simple examples like Crofton's classical result on the line-measure of a closed convex domain in the plane. After describing the fundamental formula of Santaló-Blaschke and extensions due to Hadwiger, recent applications of integral geometry to multi-dimensional information theory and toward a quantitative treatment of the problem of shape of sets (in particular in "pattern recognition") are sketched.

11. *Finite geometries and the four color problem*, by Professor W. T. Tutte, University of Toronto.

The problem of coloring a map in four colors can be reduced in various ways to that of solving a system of linear inequalities over a finite field. This approach leads to the problem of classifying those sets of points in $PG(n, 2)$ which have a member on each $(n-2)$-dimensional hyperplane. This classification problem is solved in 2 and 3 dimensions; results obtained in 4 and 5 dimensions are stated; and a conjecture, which implies the Four Color Theorem, is made for space of n dimensions.

N. G. GUNDERSON, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The thirteenth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Montana State University, Missoula, Montana, on June 17, 1960 in conjunction with the 570th meeting of the American Mathematical Society and jointly with the Pacific Northwest Section of the Society for Industrial and Applied Mathematics. Professor A. E. Livingston, Chairman of the Section, presided over the meetings. There were 35 members of the Association registered.

The business meeting was held at 8:00 P.M., June 17. The following officers were elected: Chairman, Professor T. G. Ostrom, Washington State University; Vice-Chairman, Professor R. E. Gaskell, Oregon State College; Secretary-Treasurer, Professor K. S. Ghent.

The program included an invited hour address by Professor Paul Rosenbloom, four invited half-hour talks, and a contributed paper. The abstracts for the papers are given below:

1. *Implications for the colleges of the new school mathematics programs*, by Professor P. C. Rosenbloom, University of Minnesota and Minnesota State Department of Education.

The common features of the programs of UICSM, Ball State, the Commission on Mathematics, and SMSG were described. One major aim of these programs is to make the standard freshman course in college a substantial course in analytic geometry and calculus. Actually much analytic geometry is already belng taught in high school. The new curricula place a strong emphasis on understanding rather than rote mastery. All have a goodly proportion of nonroutine problems, and all introduce deduction into algebra as well as geometry. They will produce college freshmen who are no more gifted, but much more mature mathematically than our present crop.

Invited thirty-minute talks:

2. *Topics in Boolean algebra*, by Professor J. E. Whitesitt, Montana State College.

In this expository talk, a Boolean logic was introduced as a pair (A, I), where A is a Boolean algebra (the set of propositions) and I is an ideal (the set of refutable propositions) in A. The algebraic nature of consistency and completeness were discussed with the conclusion that a Boolean logic is consistent and complete if and only if the quotient algebra A/I is isomorphic to the two element Boolean algebra $\{0, 1\}$. Finally the notion of Boolean algebra was extended to that of a monadic algebra. A monadic logic was defined and consistency and completeness for monadic logics introduced.

3. *Generalized rings*, by Professor R. A. Beaumont, University of Washington.

A generalized ring, or an (m, n)-ring is an additive abelian group R, closed with respect to multiplication, which satisfies a general distributive law $(\sum_{i=1}^{m} a_i)\cdot(\sum_{j=1}^{n} b_j)=\sum_{i,j=1}^{m,n} a_ib_j$ for a fixed pair of integers $m\geqq 2$, $n\geqq 2$, and for all a_i and b_j in R. The properties of (m, n)-rings are discussed. If I is an ideal in an (m, n)-ring R, then the difference ring $R-I$ can be formed, and it is an ordinary ring. A structure theorem for a certain class of (m, n)-rings is obtained. Every (m, n)-ring can be made into an ordinary ring by defining a new multiplication in terms of the given multiplication.

4. *Characteristics of the reliability of complex systems*, by Dr. J. D. Esary and Dr. S. C. Saunders, Boeing Scientific Research Laboratories, and Professor Z. W. Birnbaum, University of Washington, presented by Dr. Esary.

A dichotomic functional structure consists of n variables $X_i=0, 1$ (indicators for functioning of components) and a function ϕ mapping $\mathbf{X}=(X_1, \cdots, X_n)$ into $\{0, 1\}$ (indicator for functioning of system). It is assumed that $\Pr(X_i=1)=p$, $i=1, \cdots, n$, and the X_i are statistically independent. The characteristics examined are those of system reliability, $h(p)=\Pr(\phi=1\,|\,p)$, compared with component reliability p. When the structure is *coherent*, *i.e.*, $\phi(0)=0$, $\phi(1)=1$, $\phi(\mathbf{X})\geqq\phi(\mathbf{Y})$ whenever $X_i\geqq Y_i$, $i=1, \cdots, m$, then h is (with specified exceptions) S-shaped in the sense that $h(0)=0$, $h(1)=1$, $h(p)<p$ for $p<\rho$ and $h(p)>p$ for $p>\rho$ for some $\rho\in(0, 1)$. When the structure is *coherent in probability*, *i.e.*, $\Pr(\phi=1\,|\,S=0)=0$, $\Pr(\phi=1\,|\,S=n)=1$, $\Pr(\phi=1\,|\,S=k)\leqq\Pr(\phi=1\,|\,S=k+1)$, $k=0, \cdots, m-1$, where $S(\mathbf{X})=X_1+\cdots+X_n$, then $h(0)=0$, $h(1)=1$, and h is a strictly increasing function of increasing p.

5. *The computer as an object of research*, by Professors A. T. Lonseth, Oregon State College, and F. H. Young, Montana State University.

Oregon State College's present computer organization will be expanded by the construction of a large, fast computer similar to the MANIAC III at Chicago. This will be incorporated in a Computer Research Center with a staff of numerical analysts, logicians, circuit and system designers, and components specialists. Four objects of research will be: (1) construction of computer languages; (2) computer logic; (3) analysis of arithmetical schemes; (4) investigation of compo-

nents and structures. Three possible levels of concentration requiring interdepartmental cooperation are: (1) construction (mathematics, electrical engineering, physics, chemistry); (2) application (mathematics, economics, business administration, sciences); (3) implication (mathematics, philosophy, psychology). The third area has hitherto received insufficient attention.

Contributed ten minute paper:

6. *An extension of Graeffe's method*, by Professor J. P. Ballantine, University of Washington.

$P(x, h)$ is called a factornomial of degree n if it is the product of n linear factors in x and h. If $P_0(x)$ and $P_1(x)$ in the Maclaurin expansion of a factornomial, $P(x, h)=P_0(x)+hP_1(x)+\cdots$, are known, then $P(x, h)$ is unique. $G(P(x, h))$ is defined as the unique factornomial determined by $P_0(s)P_0(-s)+\frac{1}{2}h[P_0(s)P_1(-s)+P_0(-s)P_1(s)]$, where $s^2=x$. An x-root of $P(x, h)$ is any number x_i such that $P(x_i, 0)=0$. An h-root is any number such that $P(0, h_i)=0$. Under G, x-roots are squared and h-roots remain unchanged. To find the roots of $P(x)$, find the h-roots of the factornomial $P(x+h)$, which are the same as the h-roots of $G^kP(x+h)$. The latter is factored by inspection, as in Graeffe's method, whence the h-roots are easily found.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA

IOWA, Simpson College, Indianola, April 21, 1961.

KANSAS

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, December 3, 1960.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA, University of North Dakota, Grand Forks, November 5, 1960.

MISSOURI, University of Missouri, Columbia Spring, 1961,

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 5, 1960.

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, University of Colorado, Boulder, May, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephan F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.

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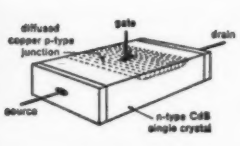

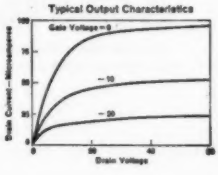

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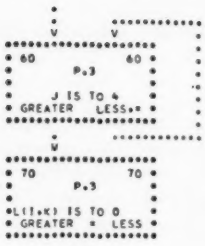

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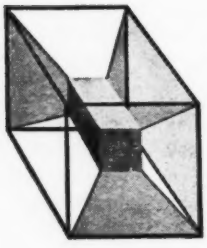

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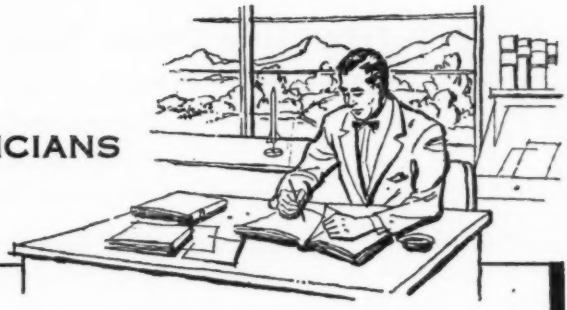

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